

TOWARDS MIRROR SYMMETRY FOR VARIETIES OF GENERAL TYPE

MARK GROSS, LUDMIL KATZARKOV, HELGE RUDDAT

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INTRODUCTION

The main goal of this paper is to propose a theory of mirror symmetry for varieties of general type. At first glance, the existence of such a theory would perhaps seem unlikely. After all, if S, \check{S} were a mirror pair with S of general type and dimension d , and if the first symptom of mirror symmetry is a reflection of the Hodge diamond, then we must face the possibility of having, say, $h^{0,0}(\check{S}) = h^{d,0}(S)$ being larger than 1. So it is clear that the mirror \check{S} should not be a variety in the ordinary sense.

In this paper we will propose that the mirror to a variety of general type is a reducible variety equipped with a certain perverse sheaf. The cohomology of this perverse sheaf will carry a mixed Hodge structure which we expect has the desired features.

The motivation for these structures arises from the study of Landau-Ginzburg models, i.e., pairs (X, w) with X a variety and $w : X \rightarrow \mathbb{C}$ a non-constant regular function. Let

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us consider a very basic form of mirror symmetry, involving duality between cones. Set $M \cong \mathbb{Z}^n$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Consider a strictly convex rational polyhedral cone $\sigma \subseteq M_{\mathbb{R}}$ with $\dim \sigma = \dim M_{\mathbb{R}}$, and let $\check{\sigma} \subseteq N_{\mathbb{R}}$ be the dual cone,

$$\check{\sigma} := \{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geq 0 \text{ for all } m \in \sigma\}.$$

The corresponding toric varieties

$$X_{\sigma} := \text{Spec } \mathbb{C}[\check{\sigma} \cap N]$$

$$X_{\check{\sigma}} := \text{Spec } \mathbb{C}[\sigma \cap M]$$

are usually singular. Choose desingularizations by choosing fans Σ and $\check{\Sigma}$ which are refinements of σ and $\check{\sigma}$ respectively, with Σ and $\check{\Sigma}$ consisting only of standard cones, i.e., cones generated by part of a basis for M or N .

We now obtain smooth toric varieties X_{Σ} and $X_{\check{\Sigma}}$, and in addition, we obtain Landau-Ginzburg potentials as follows. For each ray $\rho \in \Sigma$, let $m_{\rho} \in M$ be the primitive generator of ρ , so that $z^{m_{\rho}}$ is a monomial regular function on $X_{\check{\Sigma}}$. Similarly, for each ray $\check{\rho} \in \check{\Sigma}$, with primitive generator $n_{\check{\rho}} \in N$, $z^{n_{\check{\rho}}}$ is a monomial function on X_{Σ} . We then define Landau-Ginzburg potentials $w : X_{\Sigma} \rightarrow \mathbb{C}$ and $\check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$ as

$$(0.1) \quad w := \sum_{\check{\rho}} c_{\check{\rho}} z^{n_{\check{\rho}}}$$

$$(0.2) \quad \check{w} := \sum_{\rho} c_{\rho} z^{m_{\rho}}$$

where $c_{\check{\rho}}, c_{\rho} \in \mathbb{C}$ are general coefficients. Note w (resp. \check{w}) factors through the resolution $X_{\Sigma} \rightarrow X_{\sigma}$ (resp. $X_{\check{\Sigma}} \rightarrow X_{\check{\sigma}}$).

Now in general, it is currently understood that given a Landau-Ginzburg model $w : X \rightarrow \mathbb{C}$, the correct cohomology group to associate to this model is the one obtained from the *twisted de Rham complex*, see [KKP08], 3.2. In this context, since in general w is not proper, we need to partially compactify first. We choose a partial compactification $X \subseteq \bar{X}$ with $D := \bar{X} \setminus X$ being normal crossings and such that w extends to a projective map $\bar{w} : \bar{X} \rightarrow \mathbb{C}$. We then consider the complex

$$(\Omega_{\bar{X}}^{\bullet}(\log D)[u], u d + d\bar{w} \wedge)$$

where $u \in \mathbb{C}$ is a parameter. The relevant cohomology groups in the Landau-Ginzburg theory are the hypercohomology groups of this complex. By a theorem of Barannikov and Kontsevich (unpublished), the hypercohomology is a free $\mathbb{C}[u]$ -module. New proofs were given by Sabbah [Sab99] and Ogus and Vologodsky [OV07]. As stated in [Sab99], this result takes the following form:

Theorem 0.1. *Let $\bar{w} : \bar{X} \rightarrow \mathbb{C}$ be projective, $D \subseteq \bar{X}$ a normal crossing divisor. Then*

(1) *The hypercohomology groups of the complexes*

$$(\Omega_{\bar{X}}^{\bullet}(\log D), d + d\bar{w}\wedge) \quad \text{and} \quad (\Omega_{\bar{X}}^{\bullet}(\log D), d\bar{w}\wedge)$$

have the same dimensions.

(2) *Let $p_1, \dots, p_k \in \mathbb{C}$ be the critical values for \bar{w} and $j : X = \bar{X} \setminus D \hookrightarrow \bar{X}$ the inclusion. Then in the analytic topology,*

$$(0.3) \quad \dim \mathbb{H}^i(\bar{X}, (\Omega_{\bar{X}}^{\bullet}(\log D), d\bar{w}\wedge)) = \sum_{j=1}^k \dim \mathbb{H}^{i-1}(\bar{w}^{-1}(p_j), \phi_{\bar{w}, p_i} \mathbf{R}j_* \mathbb{C}_X).$$

Here $\phi_{\bar{w}, p_i}$ denotes the vanishing cycle functor at p_i , for a precise definition, see §4. Most importantly for our current discussion, $\phi_{\bar{w}, p_i} \mathbf{R}j_* \mathbb{C}_X$ is a sheaf supported on the critical locus of each singular fibre.

Now one subtlety in Landau-Ginzburg mirror symmetry is that this cohomology group is often too big, because there are some singular fibres of w which we do not wish to consider. These singular fibres often “come in from infinity” as the Landau-Ginzburg potential is varied, and to get the correct group, we need to ignore these fibres. In particular, we should only use certain singular fibres in the sum (0.3). We make suggestions in §6.3 about how to deal with this in general.

In the general setup of X_{Σ} , $X_{\tilde{\Sigma}}$ given above, the singular fibres can be quite complicated. We are not yet proposing a general method for computing the relevant cohomology groups in this setup. On the other hand, we may restrict attention to a special case for which we show a mirror duality of Hodge numbers.

So we now specialize to the following setup.

Let $\Delta \subseteq M_{\mathbb{R}}$ be a lattice polytope which is the Newton (moment) polytope of a non-singular projective toric variety \mathbb{P}_{Δ} . Define the cone $\text{Cone}(\Delta) \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$ by

$$\text{Cone}(\Delta) := \{(rm, r) | m \in \Delta, r \geq 0\}.$$

We can take $\sigma = \text{Cone}(\Delta)$ in the above construction. We now subdivide σ by choosing a triangulation of Δ into standard simplices; we assume that we can do this. This then gives rise to a fan Σ consisting of cones over these simplices. Geometrically, X_{Σ} is a crepant resolution of the Gorenstein singularity X_{σ} . On the other hand, as we shall check in §1, the cone $\tilde{\sigma}$ can be subdivided to give a fan $\tilde{\Sigma}$ via a star subdivision with center the ray generated by $(0, 1) \in N_{\mathbb{R}} \oplus \mathbb{R}$. Geometrically, this is the contraction of the zero section

$$X_{\tilde{\Sigma}} = \text{Tot}(\mathcal{O}_{\mathbb{P}_{\Delta}}(-1)) \rightarrow X_{\tilde{\sigma}},$$

where $\text{Tot}(\mathcal{V})$ denotes the relative Spec of $\text{Sym } \mathcal{V}^*$ for a vector bundle \mathcal{V} .

Given these choices of X_{Σ} and $X_{\tilde{\Sigma}}$, we obtain as above Landau-Ginzburg potentials w and \tilde{w} on these two spaces respectively. As we shall see in §1, the origin $0 \in \mathbb{C}$ is a critical value for both w and \tilde{w} . Furthermore, $\tilde{w}^{-1}(0)$ is quite simple, consisting of a

normal crossings union of two divisors whose intersection is a hyperplane section S of \mathbb{P}_Δ determined by \check{w} . One can show that $\phi_{\check{w},0}\mathbf{R}j_*\mathbb{C}_{X_{\check{\Sigma}}}$ is just the constant sheaf \mathbb{C} on S (shifted in degree). Moreover, the derived category of coherent sheaves on S is equivalent to the category of singularities of $\check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$ by a generalized Knörrer periodicity, see §2.1. In particular since 0 is the only critical value of \check{w} ,

$$\mathbb{H}^i(\bar{X}_{\check{\Sigma}}, (\Omega_{\bar{X}_{\check{\Sigma}}}^\bullet, d\check{w}\wedge)) = \mathbb{H}^{i-1}(\check{w}^{-1}(0), \phi_{\check{w},0}\mathbf{R}j_*\mathbb{C}_{X_{\check{\Sigma}}}) = H^{i-2}(S, \mathbb{C}).$$

On the other hand, $w^{-1}(0)$ will, in general, be much more complicated, although with the assumptions given, it will still be normal crossings as long as S has non-negative Kodaira dimension $\kappa(S)$. Most of the time we will restrict ourselves to this case because the case of negative Kodaira dimension behaves somewhat differently and has already been studied intensively in the literature. See §7 for a detailed discussion of a non-trivial negative Kodaira dimension example. So let us assume $\kappa(S) \geq 0$. The singular locus of $w^{-1}(0)$ is in fact proper (before compactifying) and the perverse sheaf $\phi_{w,0}\mathbf{R}j_*\mathbb{C}_{X_\Sigma}$ is supported entirely on this singular locus.

Our proposal for addressing to a first approximation the question raised at the beginning of the paper is as follows. Let \check{S} be the singular locus of $w^{-1}(0)$, and let

$$(0.4) \quad \mathcal{F}_{\check{S}} := \phi_{w,0}\mathbf{R}j_*\mathbb{C}_{X_\Sigma}[1].$$

Then we should consider the pair $(\check{S}, \mathcal{F}_{\check{S}})$ to be mirror to the pair $(S, \underline{\mathbb{C}})$, where $\underline{\mathbb{C}}$ denotes the constant sheaf on S with coefficients \mathbb{C} .

This should give a version of mirror symmetry for which we verify the symmetry of Hodge numbers as follows. Note that $\phi_{\check{w},0}\mathbf{R}j_*\mathbb{C}_{X_{\check{\Sigma}}}$ supports a mixed Hodge structure given by Schmid-Steenbrink. We transport this to $\mathcal{F}_{\check{S}}$ using (0.4) and apply the shift $[1]$ to the Hodge and also to the weight filtration, i.e.,

$$h^{p,q}\mathbb{H}^i(\check{S}, \mathcal{F}_{\check{S}}) = h^{p+1,q+1}\mathbb{H}^{i+1}(w^{-1}(0), \phi_{w,0}\mathbf{R}j_*\mathbb{C}_X).$$

It can be observed that the weight filtration reflects the Kodaira dimension of S , see Prop. 5.1. However, we shall discard the weights and define

$$(0.5) \quad h^{p,q}(\check{S}, \mathcal{F}_{\check{S}}) = \sum_k h^{p,q+k} \mathbb{H}^{p+q}(\check{S}, \mathcal{F}_{\check{S}}).$$

Our main theorem is then:

Theorem 0.2. *Assume that the fan Σ comes from a star-like decomposition of Δ (see Def. 1.3). Then $h^{p,q}(S) = h^{d-p,q}(\check{S}, \mathcal{F}_{\check{S}})$, with $d = \dim S$.*

Note that this result implies that $h^{p,q}(\check{S}, \mathcal{F}_{\check{S}})$ is independent of the choice of a crepant resolution $X_\Sigma \rightarrow X_\sigma$, i.e., independent of the choice of a triangulation of Δ . The result suggests that there might also be a version $h^{p,q}(X, w) = h^{\dim X - p, q}(\check{X}, \check{w})$; however, it is not

currently known how to define $h^{p,q}(X, w)$ and $h^{\dim X - p, q}(\check{X}, \check{w})$ directly from the twisted de Rham complex without using Thm. 0.1. Note however that rearranging indices yields:

Corollary 0.3. *With the assumption of Thm. 0.2,*

$$h^{p,q}(X_\Sigma, w) = h^{n-p,q}(X_{\check{\Sigma}}, \check{w})$$

where $n = \dim X_\Sigma$, $h^{p,q}(X_\Sigma, w) = \sum_k h^{p,q+k} H^{p+q}(X_\Sigma, w)$, and $H^i(X_\Sigma, w)$ is defined as the $(i-1)$ th hypercohomology of $\phi_{w,0} \mathbf{R}j_* \mathbf{C}_{X_\Sigma}$ with its Schmid-Steenbrink mixed Hodge structure (analogously for $(X_{\check{\Sigma}}, \check{w})$).

A formula for $h^{p,q}(S)$ was given by Danilov-Khovanskii. To compute $h^{p,q}(\check{S})$, we use the toric description of the resolution X_Σ and the weight filtration spectral sequence of the cohomological mixed Hodge complexes computing vanishing cycles.

The structure of the paper is as follows. In §1, we introduce the combinatorial setup and describe in detail the construction of the proposed Landau-Ginzburg mirrors and their structure. In §2 we speculate on homological mirror symmetry for our constructions, and explain its relationship with the results we prove in the paper about cohomology groups. This section, as well as §6, can be viewed as extensions of this introduction. §3 reviews basic formulae for Hodge numbers of hypersurfaces in toric varieties. §4 fills in some of the necessary background in mixed Hodge theory. §5 then gives the details of the calculation of the Hodge numbers of the mirror: this is the heart of the paper. §6 discusses, without too many details, various additional issues associated to our proposals: the relationship of our construction with the discrete Legendre transform and the Gross-Siebert picture; the relationship with the proposal of Abouzaid, Auroux and Katzarkov [AAK]; mirrors for complete intersections; and an orbifold version of some of our conjectures. Finally, §7 considers in detail a Fano example, namely the cubic threefold.

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1. THE SETUP: THE MIRROR PAIR OF LANDAU-GINZBURG MODELS

1.1. Resolutions. As in the introduction, let

$$M \cong \mathbb{Z}^{d+1}, \quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}, \quad N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z}), \quad N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}.$$

We will also use the notation

$$\bar{M} \cong M \oplus \mathbb{Z}, \quad \bar{M}_{\mathbb{R}} = \bar{M} \otimes_{\mathbb{Z}} \mathbb{R}, \quad \bar{N} = \operatorname{Hom}_{\mathbb{Z}}(\bar{M}, \mathbb{Z}), \quad \bar{N}_{\mathbb{R}} = \bar{N} \otimes_{\mathbb{Z}} \mathbb{R}.$$

We recall briefly the standard correspondence between convex polyhedra and fans. First, a *lattice polyhedron* $\Delta \subseteq M_{\mathbb{R}}$ is an intersection Δ of half-spaces with boundary of rational

slope such that Δ has at least one vertex and all vertices of Δ lie in M . A *lattice polytope* is a compact lattice polyhedron. If $\text{Cone}(\Delta)$ denotes the cone over Δ as defined in the introduction, then a lattice polyhedron Δ defines a toric variety \mathbb{P}_Δ by

$$\mathbb{P}_\Delta = \text{Proj } \mathbb{C}[\text{Cone}(\Delta) \cap (M \oplus \mathbb{Z})]$$

where $\mathbb{C}[P]$ denotes the monoid algebra of a monoid P . For $\tau \subseteq \Delta$ a face, the normal cone to Δ along τ is

$$N_\Delta(\tau) = \{n \in N \mid n|_\tau = \text{constant}, \langle n, m \rangle \geq \langle n, m' \rangle \text{ for all } m \in \Delta, m' \in \tau\}.$$

The normal fan of Δ is

$$\check{\Sigma}_\Delta := \{N_\Delta(\tau) \mid \tau \text{ is a face of } \Delta\}.$$

The normal fan $\check{\Sigma}_\Delta$ carries a strictly convex piecewise linear function φ_Δ defined by

$$\varphi_\Delta(n) = -\inf\{\langle n, m \rangle \mid m \in \Delta\}.$$

Conversely, given a fan Σ in $N_\mathbb{R}$ whose support $|\Sigma|$ is convex, and given a strictly convex piecewise linear function with integral slopes $\varphi : |\Sigma| \rightarrow \mathbb{R}$, the *Newton polyhedron* of φ is

$$\Delta_\varphi := \{m \in M_\mathbb{R} \mid \varphi(n) + \langle n, m \rangle \geq 0 \text{ for all } n \in |\Sigma|\}.$$

By standard toric geometry this coincides up to translation with the convex hull of all points of M indexing monomial sections of the line bundle associated to the divisor $\sum_\rho \varphi(n_\rho) D_\rho$. Here the sum is taken over the rays ρ of Σ , D_ρ being the corresponding toric prime divisor, and n_ρ the primitive generator of ρ . So we may also associate a Newton polytope to a Laurent polynomial or a line bundle.¹

If Σ is a fan, we denote by X_Σ the toric variety defined by Σ . If σ is a strictly convex rational polyhedral cone, then we write X_σ for the affine toric variety defined by the cone σ . Given $\tau \in \Sigma$, $V(\tau)$ will denote the closure of the torus orbit in X_Σ corresponding to τ , e.g., $V(\{0\}) = X_\Sigma$. For $\rho \in \Sigma$ a ray, $V(\rho)$ is a toric divisor which we will also call D_ρ .

Now fix once and for all a lattice polytope $\Delta \subseteq M_\mathbb{R}$ with $\dim \Delta = \dim M_\mathbb{R} > 0$. This defines a projective toric variety \mathbb{P}_Δ with an ample line bundle $\mathcal{O}_{\mathbb{P}_\Delta}(1)$ with Newton polytope Δ . The fan defining this toric variety is the normal fan $\check{\Sigma}_\Delta$ of Δ , and the line bundle $\mathcal{O}_{\mathbb{P}_\Delta}(1)$ is induced by the piecewise linear function $\varphi_\Delta : N_\mathbb{R} \rightarrow \mathbb{R}$ on the fan $\check{\Sigma}_\Delta$.

We shall assume throughout this paper that \mathbb{P}_Δ is a non-singular variety. This is equivalent to each cone in the normal fan to Δ being a standard cone, i.e., being generated by e_1, \dots, e_i , where e_1, \dots, e_{d+1} is a basis of N . We shall also assume that Δ has at least one interior integral point. As we will see in §1.3, this is equivalent to the condition $\kappa(S) \geq 0$ used in the introduction.

¹If no global section exists, the polytope will be empty.

We define $\sigma = \text{Cone}(\Delta) \subseteq \bar{M}_{\mathbb{R}}$ as in the introduction and $\check{\sigma} = \sigma^{\vee} \subseteq \bar{N}_{\mathbb{R}}$ where the dual cone is defined by

$$\sigma^{\vee} = \{n \in \bar{N}_{\mathbb{R}} \mid \langle m, n \rangle \geq 0 \text{ for all } m \in \bar{M}_{\mathbb{R}}\}.$$

Note that

$$\check{\sigma} = \{(n, r) \mid r \geq \varphi_{\Delta}(n)\}.$$

Our first task is to specify precisely the subdivisions of the cones σ and $\check{\sigma}$ we will use. There is a canonical choice of resolution for $\check{\sigma}$:

Proposition 1.1. *Let $\rho := (0, 1) \in \bar{N}_{\mathbb{R}}$. Then $\rho \in \text{Int}(\check{\sigma})$. Furthermore, let $\check{\Sigma}$ be the fan given by*

$$\begin{aligned} \check{\Sigma} := & \{ \check{\tau} \mid \check{\tau} \text{ a proper face of } \check{\sigma} \} \\ & \cup \{ \check{\tau} + \mathbb{R}_{\geq 0}\rho \mid \check{\tau} \text{ a proper face of } \check{\sigma} \}. \end{aligned}$$

This is the star subdivision of the cone $\check{\sigma}$ along the ray $\mathbb{R}_{\geq 0}\rho$. Then $X_{\check{\Sigma}}$ is a non-singular variety.

Proof. The first statement is obvious, since ρ is strictly positive on every element of $\text{Cone}(\Delta) \setminus \{0\}$. For the fact that $X_{\check{\Sigma}}$ is non-singular, let $\check{\tau}$ be a proper face of $\check{\sigma}$. Then $\check{\tau}$ takes the form

$$\check{\tau} = \{(n, \varphi_{\Delta}(n)) \mid n \in \check{\tau}'\}$$

for some $\check{\tau}' \in \check{\Sigma}_{\Delta}$. In particular, since \mathbb{P}_{Δ} is assumed to be non-singular, $\check{\tau}'$ is a standard cone, say generated by e_1, \dots, e_i , part of a basis. Then $\check{\tau} + \mathbb{R}_{\geq 0}\rho$ is generated by $(e_1, \varphi_{\Delta}(e_1)), \dots, (e_i, \varphi_{\Delta}(e_i)), (0, 1)$ which extends to a basis of \bar{N} . \square

Remark 1.2. Note that the projection $\bar{N} \rightarrow N$ induces a map on fans from $\check{\Sigma}$ to $\check{\Sigma}_{\Delta}$, so we have a morphism $X_{\check{\Sigma}} \rightarrow \mathbb{P}_{\Delta}$. This is clearly an \mathbb{A}^1 -bundle, and the source is the total space of $\mathcal{O}_{\mathbb{P}_{\Delta}}(-1)$.

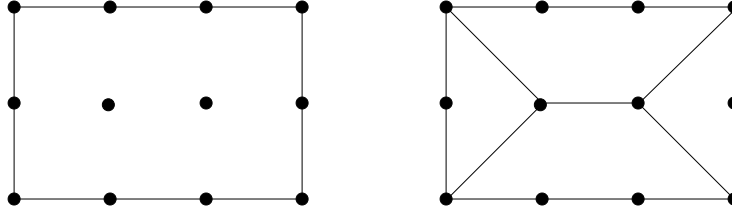
Next, we will describe allowable refinements of σ . As we see shortly, we will only consider those corresponding to crepant resolutions, i.e., refinements which arise from polyhedral decompositions \mathcal{P} of Δ into lattice polytopes. We first give a canonically determined polyhedral decomposition of Δ .

Let $h_* : \Delta \cap M \rightarrow \mathbb{Z}$ be the function defined by

$$h_*(m) = \begin{cases} 0 & \text{if } m \in \partial\Delta \\ -1 & \text{if } m \in \text{Int}(\Delta) \end{cases}$$

and

$$(1.1) \quad \Delta_* := \text{Conv}\{(m, h_*(m)) \mid m \in \Delta \cap M\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

FIGURE 1. A polytope Δ and its subdivision \mathcal{P}_*

Here $\text{Conv } A$ denotes the convex hull of a set A . Then Δ_* has one face (the upper face) equal to $\Delta \times \{0\}$, and the remaining proper faces define, via projection to $M_{\mathbb{R}}$, a subdivision of Δ . Let \mathcal{P}_* denote the set of faces of this subdivision.

Definition 1.3. A polyhedral decomposition \mathcal{P} of Δ is said to be *star-like* if it is a regular² refinement of \mathcal{P}_* .

We will assume from now on the existence of the following:

Assumption 1.4. Let \mathcal{P} be a star-like triangulation of Δ into standard simplices, i.e., simplices τ such that $\text{Cone}(\tau)$ is a standard cone.

Such a triangulation need not exist; it does, however, always exist if $\dim \Delta = 2$. The existence of \mathcal{P} is equivalent to the existence of a toric crepant resolution of the blow-up of X_σ at the origin. To get rid of the Assumption 1.4, one may work with toric stacks. There always exists a crepant resolution as a toric Deligne-Mumford stack whose coarse moduli space has at worst terminal quotient singularities. Such is given by a triangulation of \mathcal{P}_* by elementary simplices, i.e., simplices whose only lattice points are its vertices. In this paper we stick to Assumption 1.4 to avoid having to develop the relevant theory on stacks. More generally, one should conjecturally use an orbifold twisted de Rham complex, orbifold cohomology and vanishing cycles on orbifolds to obtain more general results, see §6.5. Note that there are typically several choices for \mathcal{P} . These are related by “phase transitions in the Kähler moduli space.” More precisely, each choice is given by a maximal cone in the secondary fan of σ . As we will see, the Hodge numbers don’t depend on this choice.

Having fixed \mathcal{P} , we obtain a refinement Σ of σ by

$$\Sigma = \{\text{Cone}(\tau) \mid \tau \in \mathcal{P}\} \cup \{\{0\}\}$$

and similarly Σ_* replacing \mathcal{P} by \mathcal{P}_* . Geometrically, we have a composition

$$X_\Sigma \rightarrow X_{\Sigma_*} \rightarrow X_\sigma$$

where the second map is the blow-up of the origin in X_σ ; this will be explained in §1.4.

²Recall a polyhedral decomposition \mathcal{P} of Δ is *regular* if there is a strictly convex piecewise linear function on Δ whose maximal domains of linearity are the cells of \mathcal{P} .

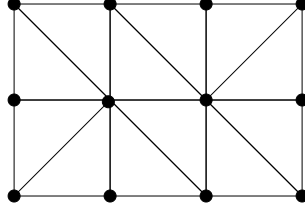


FIGURE 2. A star-like subdivision giving a crepant resolution

Example 1.5. Let Δ be a reflexive polytope, i.e.,

- a) Δ has a unique interior lattice point v and
- b) the polar dual $\Delta^* := \{n \in N_{\mathbb{R}} \mid \langle n, m - v \rangle \geq -1 \quad \forall m \in \Delta\}$ is a lattice polytope.

Under Assumption 1.4, a) implies b). It is not hard to see that $\check{\sigma} = \text{Cone}(\Delta)^\vee = \text{Cone}(\Delta^*)$. In this case, \mathcal{P}_* is the star subdivision of Δ at v . This is the subdivision whose maximal cells are the convex hulls of $\tau \cup \{v\}$ with τ a maximal proper face of Δ .

Example 1.6. This will be a running example throughout the paper. We consider the two-dimensional polytope drawn on the left in Figure 1. The picture on the right gives \mathcal{P}_* . We then have several possible choices for \mathcal{P} ; for example, we may take the one given in Figure 2.

We can now choose Landau-Ginzburg potentials

$$\begin{aligned} w &: X_\Sigma \rightarrow \mathbb{C}, \\ \check{w} &: X_{\check{\Sigma}} \rightarrow \mathbb{C}. \end{aligned}$$

We write these as follows. First, for w , the primitive generators of one-dimensional cones of $\check{\Sigma}$ are $\rho = (0, 1) \in N \oplus \mathbb{Z}$ and $(n_\tau, \varphi_\Delta(n_\tau))$, where τ runs over codimension one faces of Δ and n_τ is the primitive (inward-pointing) normal vector to τ . Thus we write

$$(1.2) \quad w = c_\rho z^\rho + \sum_{\tau \subset \Delta} c_\tau z^{(n_\tau, \varphi_\Delta(n_\tau))},$$

where again the sum is over all codimension one faces of Δ . Second, the primitive generators of the one-dimensional cones of Σ are of the form $(m, 1)$ for $m \in \Delta \cap M$, so we write

$$(1.3) \quad \check{w} = \sum_{m \in \Delta \cap M} c_m z^{(m, 1)}.$$

Here all coefficients are chosen in \mathbb{C} generally. In Prop. 1.8, we note that giving \check{w} is equivalent to giving a global section of $\mathcal{O}_{\mathbb{P}_\Delta}(1)$ and show that its zero locus S coincides with the critical locus of \check{w} .

Example 1.7. Continuing and extending Ex. 1.6, we may take for Δ a rectangle of edge lengths 2 and $g + 1$ such that Δ has g interior points and S is a genus g curve. Before the

resolution, its mirror Landau-Ginzburg model (X_σ, w) is then given via (1.2) as

$$(X_\sigma = \text{Spec } \mathbb{C}[x, y, z, u, v]/(xy - z^2, uv - z^{g+1}), c_x x + c_y y + c_z z + c_u u + c_v v)$$

where $z = z^\rho$, u, v are the monomials given by the normals of the length two edges of Δ and x, y those for the length $g + 1$ edges. The singular locus of X_σ is non-compact with four irreducible components, two of which are generically curves of A_1 singularities, the other two generically curves of A_g singularities.

1.2. Properifications. Now w and \tilde{w} are not proper, so we need to choose properifications of these maps. The particular choice will turn out not to be important, as it won't affect the answer: the sheaves of vanishing cycles whose cohomology we will eventually have to compute will have proper support even before compactifying. We still need to make some choice to show that we are not losing any cohomology, however. The two functions w and \tilde{w} are dealt with separately.

Since $X_{\tilde{\Sigma}}$ is an \mathbb{A}^1 -bundle over \mathbb{P}_Δ , the obvious thing to do is to compactify $X_{\tilde{\Sigma}}$ to a \mathbb{P}^1 -bundle over \mathbb{P}_Δ .

Proposition 1.8 (Properification of \tilde{w}). *Consider $\tilde{\tilde{\Sigma}}$ given by*

$$\begin{aligned} \tilde{\tilde{\Sigma}} := & \{ \check{\tau} \mid \check{\tau} \text{ a proper face of } \check{\sigma} \} \\ & \cup \{ \check{\tau} + \mathbb{R}_{\geq 0} \rho \mid \check{\tau} \text{ a proper face of } \check{\sigma} \} \\ & \cup \{ \check{\tau} - \mathbb{R}_{\geq 0} \rho \mid \check{\tau} \text{ a proper face of } \check{\sigma} \}. \end{aligned}$$

Then

- (1) $\tilde{\tilde{\Sigma}}$ is a complete, non-singular fan containing the fan $\tilde{\Sigma}$, hence giving a projective compactification $X_{\tilde{\Sigma}} \subseteq X_{\tilde{\tilde{\Sigma}}}$. The projection $\tilde{N} \rightarrow N$ defines a map of fans from $\tilde{\tilde{\Sigma}}$ to $\tilde{\Sigma}_\Delta$, giving a morphism $X_{\tilde{\Sigma}} \rightarrow \mathbb{P}_\Delta$ which is a \mathbb{P}^1 -bundle. Let D_0 be the divisor corresponding to the ray $\mathbb{R}_{\geq 0} \rho$ and D_∞ be the divisor corresponding to the ray $-\mathbb{R}_{\geq 0} \rho$. These are sections of the projection to \mathbb{P}_Δ , hence isomorphic to \mathbb{P}_Δ .
- (2) \tilde{w} extends to a rational map $\tilde{w} : X_{\tilde{\Sigma}} \dashrightarrow \mathbb{P}^1$ which fails to be defined on a non-singular subvariety of codimension two. Blow up this subvariety to obtain $\tilde{X}_{\tilde{\Sigma}}$. Then $\tilde{X}_{\tilde{\Sigma}} \setminus X_{\tilde{\Sigma}}$ is normal crossings. Furthermore, \tilde{w} extends to give a projective morphism $\tilde{\tilde{w}} : \tilde{X}_{\tilde{\Sigma}} \rightarrow \mathbb{P}^1$.
- (3) There is a non-singular divisor \tilde{W}_0 on $\tilde{X}_{\tilde{\Sigma}}$ such that $\tilde{\tilde{w}}^{-1}(0) = D_0 \cup \tilde{W}_0$ is a normal crossings divisors, with $D_0 \cap \tilde{W}_0$ isomorphic to the hypersurface S in \mathbb{P}_Δ given by the equation $\tilde{w} = 0$. Note this makes sense as the terms in \tilde{w} are in one-to-one correspondence with points of $\Delta \cap M$, and these points form a basis for $\mathcal{O}_{\mathbb{P}_\Delta}(1)$.

Proof. (1) is standard; we leave the details to the reader.

For (2) and (3), let us begin by considering a cone of the form $\check{\tau} \pm \mathbb{R}_{\geq 0} \rho$ in $\tilde{\tilde{\Sigma}}$, where $\check{\tau}$ is a maximal proper face of $\check{\sigma}$. We know that $\check{\tau}$ is dual to $\text{Cone}(v) \subseteq \sigma$ for some vertex v of Δ .

Furthermore, $\check{\tau} \pm \mathbb{R}_{\geq 0}\rho$ is generated by vectors $(e_1, \varphi_\Delta(e_1)), \dots, (e_{d+1}, \varphi_\Delta(e_{d+1})), \pm\rho$ where $e_i \in N$ is constant on a maximal proper face of Δ containing v and $\varphi_\Delta(e_i) = -\langle e_i, v \rangle$. Thus $\mathbb{C}[\check{\tau}^\vee \cap \bar{M}] \cong \mathbb{C}[x_1, \dots, x_{d+2}]$, where x_1, \dots, x_{d+2} are the monomials associated to the dual basis to $(e_1, \varphi_\Delta(e_1)), \dots, (e_{d+1}, \varphi_\Delta(e_{d+1})), \pm\rho$.

Now if $m \in \Delta \cap M$, a monomial $z^{(m,1)}$ can then be written in terms of x_1, \dots, x_{d+2} as

$$\begin{aligned} z^{(m,1)} &= x_{d+2}^{\pm 1} \prod_{i=1}^{d+1} x_i^{\langle (e_i, \varphi_\Delta(e_i)), (m,1) \rangle} \\ &= x_{d+2}^{\pm 1} \prod_{i=1}^{d+1} x_i^{\langle e_i, m \rangle - \langle e_i, v \rangle}. \end{aligned}$$

Note also that if e_1^*, \dots, e_{d+1}^* is the dual basis to e_1, \dots, e_{d+1} , then e_1^*, \dots, e_{d+1}^* generate the tangent cone to Δ at v , so in particular $v + e_i^* \in \Delta \cap M$. Thus up to coefficients the monomials $z^{(v,1)}$ and $z^{(v+e_i^*,1)}$ appear in \check{w} and are of the form $x_{d+2}^{\pm 1}$ and $x_{d+2}^{\pm 1} x_i$ respectively. Therefore, in this affine coordinate patch, we can write

$$\check{w} = x_{d+2}^{\pm 1} \left(c_v + \sum_{i=1}^{d+1} c_{v+e_i^*} x_i + \text{higher order terms} \right).$$

Thus, for general choice of coefficients, in the affine open subset of $X_{\check{\Sigma}}$ corresponding to $\check{\tau} + \mathbb{R}_{\geq 0}\rho$, $\check{w}^{-1}(0)$ is reducible, consisting of the two irreducible components given by $x_{d+2} = 0$ (which is the divisor corresponding to the ray $\mathbb{R}_{\geq 0}\rho$, i.e., D_0) and the hypersurface given by

$$(1.4) \quad c_v + \sum_{i=1}^{d+1} c_{v+e_i^*} x_i + \text{higher order terms} = 0.$$

Again, for general choice of coefficients, this will be non-singular.

Similarly, in the affine open subset of $X_{\check{\Sigma}}$ corresponding to $\check{\tau} - \mathbb{R}_{\geq 0}\rho$, we see that \check{w} has a simple pole along the divisor $x_{d+1} = 0$ (the divisor D_∞) and is zero along a hypersurface defined by the same equation (1.4).

Let \check{W}_0 be the closure in $X_{\check{\Sigma}}$ of the hypersurface given by (1.4) in any of the affine subsets considered. Then \check{w} is zero along $D_0 \cup \check{W}_0$ and has a simple pole along D_∞ , and \check{w} is undefined along $\check{W}_0 \cap D_\infty$.

Furthermore, the equation (1.4) restricted to either D_0 or D_∞ yields (an affine piece of) the hypersurface in \mathbb{P}_Δ defined by $\check{w} = 0$. Thus in particular, $\check{W}_0 \cap D_\infty$ is a non-singular variety of codimension two, which we may blow up to get a non-singular variety $\tilde{X}_{\check{\Sigma}}$, with exceptional hypersurface E , and \check{w} extends to a well-defined function on $\tilde{X}_{\check{\Sigma}}$. Note the proper transforms of D_0, D_∞ and \check{W}_0 in $\tilde{X}_{\check{\Sigma}}$ are isomorphic to D_0, D_∞ and \check{W}_0 , so we continue to use the same notation.

The center of the blow-up is contained in $D_\infty = X_{\tilde{\Sigma}} \setminus X_\Sigma$, so $X_{\tilde{\Sigma}}$ is an open subset of $\tilde{X}_{\tilde{\Sigma}}$, with $\tilde{X}_{\tilde{\Sigma}} \setminus X_\Sigma = D_\infty \cup E$. We have now shown (2). Then (3) follows also from the above discussion. \square

Let $\Delta' \subseteq \Delta$ be given by

$$\Delta' := \text{Conv}\{v \in \text{Int}(\Delta) \cap M\}.$$

Recall that we assume $\dim \Delta' \geq 0$.

Remark 1.9. In classical Calabi-Yau mirror symmetry as referred to in Ex. 1.5, one considers a family of hypersurfaces given as

$$0 = t \left(\sum_{m \in \Delta \cap M} c_m z^m \right) + z^v$$

where t varies, Δ is reflexive and v is its unique interior integral point, see [Bat94]. Replacing z^m by $z^{(m,1)}$, we may view this as a family of potentials

$$\check{w}_t = t\check{w} + \check{w}_0.$$

In fact this generalizes to our more general setup if we set

$$\check{w}_0 = \sum_{m \in \Delta' \cap M} c_m z^{(m,1)}$$

because in the Calabi-Yau case $\Delta' = v$. However, whereas in the Calabi-Yau case this gives a toric degeneration of the fibre $\check{w}_t = 0$ as $t \rightarrow 0$, in general it will only be a partial toric degeneration, i.e., $\check{w}_0 = 0$ consists of the union of all toric divisors in X_Σ plus one non-toric divisor given by the Laurent polynomial \check{w}_0 . We will see in §1.3 that we have the linear equivalence

$$S \sim S' - K_{\mathbb{P}_\Delta}$$

where S' is the zero locus of $\sum_{m \in \Delta' \cap M} c_m z^m$, so $S' = 0$ in the Calabi-Yau case.

We next consider the properification of $w : X_\Sigma \rightarrow \mathbb{C}$. To do this, we first consider the obvious choice of a projective toric variety on which w can be viewed as the section of a line bundle. Let

$$\check{\Delta} = \text{Conv}\{0, \rho\} \cup \{(n_\omega, \varphi_\Delta(n_\omega)) \mid \omega \subseteq \Delta \text{ a codimension one face of } \Delta\}.$$

The wary reader will notice that the corresponding dual object to $\check{\Delta}$ is $\text{Conv}(\Delta \times \{1\} \cup \{0\}) \subseteq \bar{M}_{\mathbb{R}}$ rather than Δ itself because the former is the polytope supporting the pencil given by \check{w} . Because φ_Δ is convex, one sees that 0 is a vertex of $\check{\Delta}$ and the tangent cone to $\check{\Delta}$ at 0 is precisely the cone $\check{\sigma}$. Thus the normal fan $\check{\Sigma}_{\check{\Delta}}$ to $\check{\Delta}$ is a complete fan in $\bar{M}_{\mathbb{R}}$ containing the cone σ , so $\mathbb{P}_{\check{\Delta}}$ is a compactification of X_σ . The function w_σ on X_σ defined

by the same equation as the function w on X_Σ then extends to a rational function $w_{\tilde{\Delta}}$ on $\mathbb{P}_{\tilde{\Delta}}$ given by

$$w_{\tilde{\Delta}} = \frac{c_\rho z^\rho + \sum_{\tau \subset \Delta} c_\tau z^{(n_\tau, \varphi_\Delta(n_\tau))}}{z^0}.$$

Proposition 1.10 (Properification of w). *There is a projective birational morphism $\pi : \tilde{\mathbb{P}}_{\tilde{\Delta}} \rightarrow \mathbb{P}_{\tilde{\Delta}}$ such that*

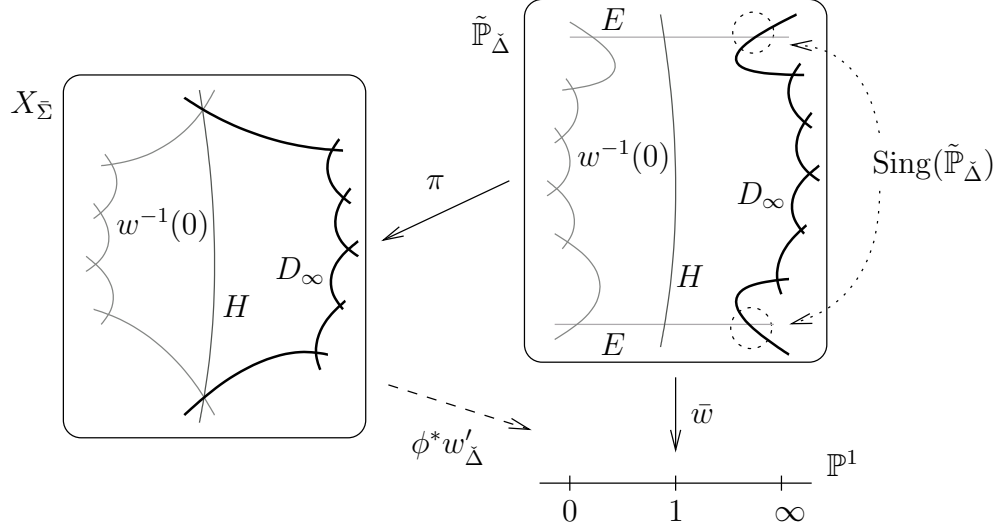
- (1) *The map π factors through a projective toric resolution of singularities $X_{\tilde{\Sigma}} \rightarrow \mathbb{P}_{\tilde{\Delta}}$ given by a fan $\tilde{\Sigma}$ which contains Σ as a subfan.*
- (2) *If $\dim \Delta' = 0$, there is a surjection $\pi_{\text{Cone}(\Delta')} : X_{\tilde{\Sigma}} \rightarrow D_{\text{Cone}(\Delta')}$ where $D_{\text{Cone}(\Delta')}$ denotes the toric divisor given by the ray $\text{Cone}(\Delta')$. The inclusion $D_{\text{Cone}(\Delta')} \rightarrow X_{\tilde{\Sigma}}$ is a section of $\pi_{\text{Cone}(\Delta')}$.*
- (3) *$\bar{w} := w_{\tilde{\Delta}} \circ \pi$ is a projective regular map to \mathbb{P}^1 .*
- (4) *$\bar{w}^{-1}(\mathbb{C})$ is non-singular, where $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$, and $X_\Sigma \subseteq \bar{w}^{-1}(\mathbb{C})$, with $D := \bar{w}^{-1}(\mathbb{C}) \setminus X_\Sigma$ a normal crossings divisor. Furthermore, $\bar{w}^{-1}(0)$ is non-singular in a neighbourhood of $\bar{w}^{-1}(0) \cap D$.*

Proof. We begin by refining the normal fan $\check{\Sigma}_{\tilde{\Delta}}$ to a fan $\tilde{\Sigma}$ with the properties

- a) $\Sigma = \{\tau \in \tilde{\Sigma} \mid \tau \subseteq \sigma\}$ and
- b) $X_{\tilde{\Sigma}}$ is a projective non-singular toric variety

as follows. Let φ_Σ denote the piecewise linear convex function giving the subdivision Σ of σ . By adding a linear function, we may assume $\varphi_\Sigma \geq 0$. Note that if one gives a function on the set of integral generators of a cone τ , there is a canonical extension to all of τ as a convex piecewise linear function. Its graph is given by the lower faces of the convex hull of the graph of the function on the set of generators. We use this construction to extend φ_Σ to all of $\bar{M}_{\mathbb{R}}$ by setting the value on a generator m of a ray contained in σ to $\varphi_\Sigma(m)$ and to zero for all further rays. One easily checks that the so-constructed functions on the cones glue such that the extension is continuous and piecewise linear. Moreover, it is convex away from $\partial\sigma$. We denote the extension by φ_Σ also. By the strict convexity of $\varphi_{\tilde{\Delta}}$ at $\partial\sigma$, for some small ϵ , we find that $\varphi_{\tilde{\Delta}} + \epsilon\varphi_\Sigma$ is a piecewise linear convex function giving a refinement of $\Sigma_{\tilde{\Delta}}$ with the property of $\tilde{\Sigma}$ in a) above. In general, this may not yet induce a desingularization, however we may refine it to such. This can be done by *pulling additional rays*, i.e., by successively inserting new rays along with star-subdivisions where each ray is generated by an integral point not contained in the support of Σ . These operations can be realized by piecewise linear functions and thus induce projective partial resolutions eventually giving a total projective resolution. We call the resulting fan $\tilde{\Sigma}$ which will be the fan in (1).

To see (2), note that we may modify the previous procedure if $\dim \Delta' = 0$ as follows. The fan of the projective toric divisor $D_{\text{Cone}(\Delta')}$ is given as the minimal fan containing the

FIGURE 3. Properification of w

maximal domains of linearity of a piecewise linear function $\bar{\varphi}'$ which we may pull back to a function φ' under the projection

$$\bar{M}_{\mathbb{R}} \rightarrow \bar{M}_{\mathbb{R}}/(\mathbb{R} \text{Cone}(\Delta')).$$

Note that φ' is piecewise linear on $\Sigma_{\tilde{\Delta}}$ because there is only one ray in $\Sigma_{\tilde{\Delta}} \setminus \Sigma$ which is in fact $-\mathbb{R}_{\geq 0} \cdot \text{Cone}(\Delta')$. We may replace $\varphi_{\tilde{\Delta}} + \epsilon\varphi_{\Sigma}$ in the above procedure by $\varphi_{\tilde{\Delta}} + \epsilon\varphi'$ to obtain a $\bar{\Sigma}$ satisfying (2).

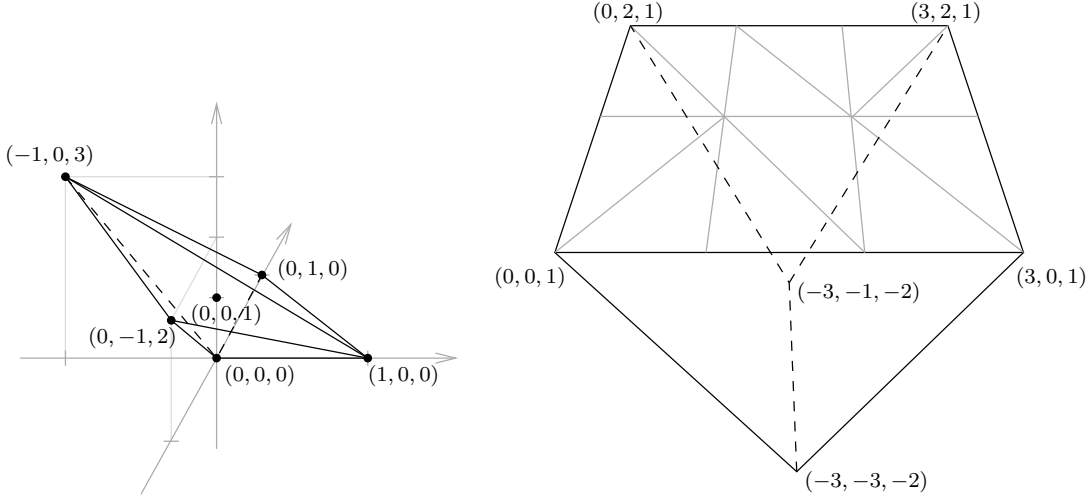
We have a resolution of singularities $\phi : X_{\bar{\Sigma}} \rightarrow \mathbb{P}_{\tilde{\Delta}}$ with $X_{\Sigma} \subseteq X_{\bar{\Sigma}}$, and since $X_{\bar{\Sigma}}$ is non-singular, $D_{\infty} := X_{\bar{\Sigma}} \setminus X_{\Sigma}$ is a divisor with normal crossings.

Next, consider the section

$$w'_{\tilde{\Delta}} = z^0 + c_{\rho} z^{\rho} + \sum_{\tau \subset \Delta} c_{\tau} z^{(n_{\tau}, \varphi_{\Delta}(n_{\tau}))}$$

of $\mathcal{O}_{\mathbb{P}_{\tilde{\Delta}}}(1)$. Because the coefficients are general, this section is $\tilde{\Delta}$ -regular in the sense of [Bat94], Def. 3.1.1. Thus pulling back this section to $X_{\bar{\Sigma}}$ we obtain a section $\phi^* w'_{\tilde{\Delta}}$ of $\phi^* \mathcal{O}_{\mathbb{P}_{\tilde{\Delta}}}(1)$ which by [Bat94], Prop. 3.2.1, is $\bar{\Sigma}$ -regular, and hence its zero locus defines a non-singular hypersurface $H \subseteq X_{\bar{\Sigma}}$. Now the rational function $w_{\tilde{\Delta}}$ pulls back to $X_{\bar{\Sigma}}$ and induces a pencil contained in the linear system $|\phi^* \mathcal{O}_{\mathbb{P}_{\tilde{\Delta}}}(1)|$. This pencil includes both the non-singular hypersurface H and the hypersurface H_{∞} given by $z^0 = 0$. One sees easily that $\text{supp}(H_{\infty}) = D_{\infty}$. Thus H_{∞} is a normal crossings divisor, but need not be reduced.

Again since H is $\bar{\Sigma}$ -regular, it meets D_{∞} transversally. So locally, at a point of $D_{\infty} \cap H$, the base-locus of the pencil defined by $w_{\tilde{\Delta}}$ on $X_{\bar{\Sigma}}$ is given by equations $x_1^{d_1} \cdots x_n^{d_n} = x_0 = 0$. Blowing up this base-locus, we obtain a projective variety $\tilde{\mathbb{P}}_{\tilde{\Delta}}$, which is singular, but now

FIGURE 4. $\check{\Delta}$ on the left and $\check{\Delta}^*$ on the right.

$w_{\check{\Delta}}$ extends to a morphism $\bar{w} : \tilde{\mathbb{P}}_{\check{\Delta}} \rightarrow \mathbb{P}^1$ factoring through the blowup map π . See Figure 3 for a picture. This gives (3). Let E be the exceptional locus of π .

Next, note from the local description of the base-locus that the singular locus of $\tilde{\mathbb{P}}_{\check{\Delta}}$ is contained entirely in $\bar{w}^{-1}(\infty)$, the proper transform of H_{∞} . Note also that X_{Σ} was disjoint from H_{∞} , and hence $X_{\Sigma} \subseteq \bar{w}^{-1}(\mathbb{C})$, the latter variety being non-singular. Furthermore, $\bar{w}^{-1}(\mathbb{C}) \setminus X_{\Sigma} = E \cap \bar{w}^{-1}(\mathbb{C})$, and from the explicit local description of $H_{\infty} \cap H$, one sees the remaining part of (4). \square

Corollary 1.11. *The morphisms $w : X_{\Sigma} \rightarrow \mathbb{C}$ and $\check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$ are quasi-projective.*

Example 1.12. Continuing with Ex. 1.6, let's assume that the vertices of Δ are $(0, 0)$, $(3, 0)$, $(0, 2)$ and $(3, 2)$. The normal fan to Δ , $\check{\Sigma}_{\Delta}$, is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$, with rays generated by $(\pm 1, 0)$ and $(0, \pm 1)$. We have

$$\varphi_{\Delta}(1, 0) = 0, \quad \varphi_{\Delta}(-1, 0) = 3, \quad \varphi_{\Delta}(0, 1) = 0, \quad \varphi_{\Delta}(0, -1) = 2$$

and hence

$$\check{\Delta} = \text{Conv}\{(0, 0, 0), (0, 0, 1), (1, 0, 0), (0, 1, 0), (-1, 0, 3), (0, -1, 2)\}$$

shown in Figure 4. One can check that the only integral points of $\check{\Delta}$ are the points listed, with $\rho = (0, 0, 1)$ the unique interior integral point of $\check{\Delta}$. So $\mathbb{P}_{\check{\Delta}}$ can be embedded in \mathbb{P}^5 using the six given points to determine sections of $\mathcal{O}_{\mathbb{P}_{\check{\Delta}}}(1)$. Using coordinates z_0, \dots, z_5 corresponding to the six points given above in the given order, one sees that the image of $\mathbb{P}_{\check{\Delta}}$ in \mathbb{P}^5 is given by the equations $z_0 z_2 z_4 - z_1^3 = 0$ and $z_3 z_5 - z_1^2 = 0$, which are homogeneous versions of those in Ex. 1.7. In addition,

$$w_{\check{\Delta}} = \frac{z_1 + z_2 + z_3 + z_4 + z_5}{z_0}.$$

Note that $\check{\Delta}$ is a reflexive polytope. This is no longer true if $g > 2$ as in Ex. 1.7.

For a general choice of $c \in \mathbb{C}$, the surface with equation $w_{\check{\Delta}} = c$ is a singular K3 surface whose inverse image \tilde{W}_c under the blowup map $X_{\tilde{\Sigma}} \rightarrow \mathbb{P}_{\check{\Delta}}$ is smooth and of Picard rank 18. The right side of Figure 4 indicates the part of the fan $\tilde{\Sigma}$ induced from the subdivision \mathcal{P} of Δ . One can view the entire fan $\tilde{\Sigma}$ by also triangulating the further faces of $\check{\Delta}^*$, but using vertices which are not necessarily integral points.

1.3. Δ' and the Kodaira dimension of S . The significance of Δ' throughout the paper is in part explained by the following results.

Proposition 1.13. *Let φ_K denote the piecewise linear function on $\check{\Sigma}_{\Delta}$ which represents $K_{\mathbb{P}_{\Delta}}$, taking the value -1 on the primitive generator of each ray of $\check{\Sigma}_{\Delta}$. Then*

$$\varphi_{\Delta'} = \varphi_{\Delta} + \varphi_K.$$

Proof. Note that φ_K exists by smoothness of \mathbb{P}_{Δ} . Let $\Delta'' = \Delta_{\varphi''}$ denote the possibly empty Newton polytope of the piecewise linear function $\varphi'' = \varphi_{\Delta} + \varphi_K$ on $\check{\Sigma}_{\Delta}$. We need to show that $\Delta'' = \Delta'$. Indeed, since Δ'' is a lattice polytope contained in the relative interior of Δ , we have $\Delta'' \subseteq \Delta'$. On the other hand $\Delta'' \supseteq \Delta'$ because, using the fact that the tangent cones to Δ at vertices of Δ are standard, each lattice point in the relative interior of Δ has integral distance ≥ 1 to each facet. \square

Corollary 1.14. *If \mathbb{P}_{Δ} has nef anti-canonical class, then the Newton polytope of $-K_{\mathbb{P}_{\Delta}}$, which we denote by Δ_K , is reflexive. We then have the Minkowski sum decomposition*

$$\Delta = \Delta_K + \Delta'.$$

Figure 5 shows this decomposition for Ex. 1.6. In the case that $-K_{\mathbb{P}_{\Delta}}$ is nef, on the dual side, we have the convex hull of the graph of $-\varphi_K$ is the cone over the dual reflexive polytope of Δ_K which we denote $\check{\Delta}_K$. This implies that

$$\check{\Delta}_K = \pi(\check{\Delta})$$

where π denotes the natural projection $\bar{N}_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$.

Now we can relate the dimension of Δ' to the Kodaira dimension of S :

Proposition 1.15. *Let S be the zero locus of a general section of $\Gamma(\mathbb{P}_{\Delta}, \mathcal{O}_{\mathbb{P}_{\Delta}}(1))$ and hence a non-singular variety of dimension d . Then the Kodaira dimension of S is*

$$\kappa(S) = \min\{\dim \Delta', d\}$$

where we use $\dim \emptyset = -\infty$.

Remark 1.16. The proposition also holds true for $\Delta' = \emptyset$ which we have excluded from our general considerations. It was pointed out to us by Victor Batyrev that smoothness of \mathbb{P}_{Δ} is necessary for the proposition to hold true because there exist hypersurfaces of general type in toric varieties with no interior lattice points in their Newton polytope.

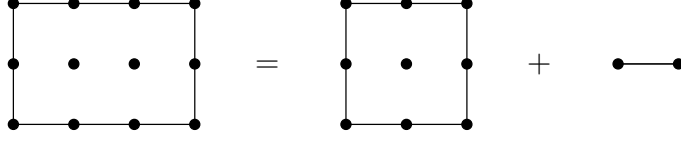


FIGURE 5.

Proof. Set $k := \min\{\dim \Delta', d\}$. We need to show that k is the minimal integer such that $\dim \Gamma(S, \mathcal{O}_S(nK_S))$ as a function of n is $O(n^k)$. Let $l(n\Delta')$ denote the number of lattice points contained in $n\Delta'$. We are done if we show that $\dim \Gamma(S, \mathcal{O}_S(nK_S)) = l(n\Delta')$ for $\dim \Delta' \leq d$ and that $\dim \Gamma(S, \mathcal{O}_S(nK_S))$ is bounded below by $l(nF)$ for $\dim \Delta' = d+1$ and some facet F of Δ' because the Kodaira dimension of S is bounded above by $\dim S = d$. By the adjunction formula, we have

$$K_S = (K_{\mathbb{P}_\Delta} + S)|_S.$$

By Proposition 1.13 and standard toric geometry, it follows that

$$l(n\Delta') = \dim \Gamma(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(n(K_{\mathbb{P}_\Delta} + S))).$$

For $\dim \Delta' \leq d$ the map $\Gamma(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(n(K_{\mathbb{P}_\Delta} + S))) \rightarrow \Gamma(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(n(K_{\mathbb{P}_\Delta} + S)) \otimes \mathcal{O}_S)$ is injective. This can be checked on the dense torus where S is given by a principal ideal a generator of which has Newton polytope Δ . Thus, every non-trivial element in the ideal has a Newton polytope of dimension $d+1$. For the same reason, for $\dim \Delta' = d+1$, the restriction of the above map to sections given by monomials in a face of Δ' is injective. \square

1.4. Geometry of the central fibre of the potential w . We now return to describing $w : X_\Sigma \rightarrow \mathbb{C}$ in more detail. In particular, we wish to describe $w^{-1}(0)$. It follows from Proposition 1.10,(4), that $\text{Sing}(w^{-1}(0))$ is proper over \mathbb{C} . We define some additional combinatorial objects. First, let

$$\check{\sigma}^o := \text{Conv}\{\bar{n} \in \bar{N} \mid \bar{n} \in \check{\sigma}, \bar{n} \neq 0\}.$$

All monomials of w lie in $\check{\sigma}^o \cap \check{\Delta}$. Moreover,

$$\left\{ \sum_{n \in I} a_n z^n \mid I \subset \check{\sigma}^o \cap \bar{N}, |I| < \infty, a_n \in \mathbb{C} \right\}$$

is the ideal of the origin 0 in X_σ . Its blow-up $\text{Bl}_0 X_\sigma$ coincides with the toric variety given by the normal fan of $\check{\sigma}^o$, see [Th03] for more details. We will see shortly that this normal fan is

$$\Sigma_* = \{\text{Cone}(\tau) \mid \tau \in \mathcal{P}_*\} \cup \{\{0\}\}$$

which we may think of as the star subdivision of σ along $\text{Cone}(\Delta')$.

We can extend the function $h_* : \Delta \cap M \rightarrow \mathbb{Z}$ to a piecewise linear function $h_* : \Delta \rightarrow \mathbb{R}$ by $h_*(m) = \inf\{r \mid (m, r) \in \Delta_*\}$ where Δ_* is defined in (1.1). This is a strictly convex

function. We now give a more useful description of \mathcal{P}_* . We recommend keeping in mind Figure 1.

Lemma 1.17. (1) *If we think of h_* as a piecewise linear function on Σ_* given by*

$$h_*(rm, r) = rh_*(m),$$

then $\check{\sigma}^o$ is the Newton polyhedron of h_ .*

(2) *We have $\Sigma_* = \check{\Sigma}_{\check{\sigma}^o}$ and*

$$X_{\Sigma_*} = \text{Bl}_0 X_\sigma = \mathbb{P}_{\check{\sigma}^o}.$$

Thus, there is a one-to-one correspondence between proper faces of $\check{\sigma}^o$ and \mathcal{P}_ which we will refer to as duality.*

(3) *Assume $\Delta' \neq \emptyset$. Then we have*

$$\mathcal{P}_* = \{\tau | \tau \subseteq \partial\Delta\} \sqcup \{\tau | \tau \in \mathcal{P}_*, \tau \not\subseteq \Delta', \tau \cap \Delta' \neq \emptyset\} \sqcup \{\tau | \tau \subseteq \Delta'\}.$$

Remark 1.18. In the language of Gross-Siebert, we have refined the discrete Legendre transform $\sigma \leftrightarrow \check{\sigma}$ to $(\Sigma_*, h_*) \leftrightarrow (\check{\sigma}^o)$. This corresponds to a blow-up X_σ and a degeneration of $X_{\check{\sigma}}$. We will come back to this point of view in §6.

Proof. Define $h'_* : \Delta \rightarrow \mathbb{R}$ by

$$h'_*(m) = - \inf_{\bar{n} \in \check{\sigma}^o} \langle \bar{n}, (m, 1) \rangle;$$

this is also a convex piecewise linear function.

To prove (1), we need to show that in fact $h_* = h'_*$. To see this, first note that for $m \in \partial\Delta \cap M$, there exists an $\bar{n} \in \check{\sigma}^o \cap \bar{N}$ such that $\langle \bar{n}, (m, 1) \rangle = 0$. Since $\langle \bar{n}', (m, 1) \rangle \geq 0$ for all $\bar{n}' \in \check{\sigma}$, we have $h'_*(m) = 0$. If $m \in \text{Int}(\Delta) \cap M$, then $\langle \rho, (m, 1) \rangle = 1$, while $\langle \bar{n}, (m, 1) \rangle \geq 1$ for all $\bar{n} \in \check{\sigma}^o \cap \bar{N}$, so $\langle \bar{n}, (m, 1) \rangle \geq 1$ for all $\bar{n} \in \check{\sigma}^o$. Thus $h'_*(m) = -1$. Now by construction, h_* is clearly the largest convex function with these values on integral points, so $h'_*(m) \leq h_*(m)$ for all $m \in \Delta$.

On the other hand, suppose $\omega \in \mathcal{P}_*$ is a maximal cell; then $h_*|_\omega$ is represented by some $\bar{n}_\omega \in N_\mathbb{R} \oplus \mathbb{R}$ on ω , identifying ω with $\omega \times \{1\} \subseteq \bar{M}_\mathbb{R}$. By Assumption 1.4, ω contains a standard simplex, and hence the integral points of $\text{Cone}(\omega) \cap (M \times \{1\})$ span \bar{M} . Since h_* only takes integral values on points of $\Delta \cap M$, we conclude that in fact \bar{n}_ω is integral, i.e., $\bar{n}_\omega \in \bar{N}$. Now we observe that $-\bar{n}_\omega \in \check{\sigma}^o$, as $\bar{n}_\omega \neq 0$ and $0 \geq h_*(m) \geq \langle \bar{n}_\omega, (m, 1) \rangle$ for all $m \in \Delta$. So for $m \in \omega$, $h'_*(m) \geq -\langle -\bar{n}_\omega, (m, 1) \rangle = h_*(m)$. Thus $h_* = h'_*$.

Because h_* is strictly convex on Σ_* , we have $\Sigma_* = \check{\Sigma}_{\check{\sigma}^o}$ and the remainder of (2) follows from what we discussed before the lemma.

Part (3) follows from the construction of h_* which makes \mathcal{P}_* be the star subdivision of Δ centered at Δ' . \square

We now refine part (3) of the previous lemma and also prove some combinatorial facts that we need later.

Lemma 1.19. (1) *If $\check{\Sigma}_{\Delta'}$ denotes the normal fan of Δ' in $N_{\mathbb{R}}/\Delta'^{\perp}$, the projection*

$$N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\Delta'^{\perp}$$

induces a map of fans

$$\check{p}_{\Delta\Delta'} : \check{\Sigma}_{\Delta} \rightarrow \check{\Sigma}_{\Delta'}.$$

(2) *There are natural maps*

$$\{\tau | \tau \subseteq \partial\Delta\} \xrightarrow{p_{\Delta\Delta'}^1} \{\tau | \tau \in \mathcal{P}_*, \tau \not\subseteq \Delta', \tau \cap \Delta' \neq \emptyset\} \xrightarrow{p_{\Delta\Delta'}^2} \{\tau | \tau \subseteq \Delta', \dim \tau < \dim \Delta\}.$$

Here $p_{\Delta\Delta'}^1$ is bijective and takes $\tau \subseteq \partial\Delta$ to the unique cell τ' of \mathcal{P}_ with $\tau' \not\subseteq \Delta'$, $\tau' \cap \Delta' \neq \emptyset$, and $\tau' \cap \partial\Delta = \tau$. The map $p_{\Delta\Delta'}^2$ is surjective and takes τ' to $\tau' \cap \Delta'$.*

We define

$$p_{\Delta\Delta'} : \{\tau | \tau \subseteq \Delta\} \rightarrow \{\tau | \tau \subseteq \Delta'\}$$

to be the composition $p_{\Delta\Delta'}^2 \circ p_{\Delta\Delta'}^1$ on proper faces of Δ , and $p_{\Delta\Delta'}(\Delta) = \Delta'$. Explicitly, for $\tau \subseteq \Delta$,

$$\begin{aligned} p_{\Delta\Delta'}(\tau) &= \text{Conv}\{p_{\Delta\Delta'}(v) | v \text{ is a vertex of } \tau\} \\ &= \Delta' \cap \bigcap_{i=1}^k \{m \in M_{\mathbb{R}} | \langle m, n_{\omega_i} \rangle = -\varphi_{\Delta}(n_{\omega_i}) + 1\} \end{aligned}$$

where ω_i are the maximal proper faces of Δ containing τ . We have $\dim \tau \geq \dim p_{\Delta\Delta'}(\tau)$. Moreover, $\check{p}_{\Delta\Delta'}$ is the composition of $p_{\Delta\Delta'}$ with the bijections which identify the set of faces of Δ , respectively Δ' , with the corresponding normal fan.

(3) *The intersection of $\check{\sigma}^{\circ}$ with $\check{\Sigma}$ induces a subdivision $\mathcal{P}_{\partial\check{\sigma}^{\circ}}$ of $\partial\check{\sigma}^{\circ}$ where each bounded face is a standard simplex. Moreover, under the duality of Lemma 1.17,(2), at most faces dual to $\tau' \subseteq \Delta'$ receive a refinement. For $\tau' \subseteq \Delta'$ and $\check{\tau}' \subseteq \check{\sigma}^{\circ}$ the corresponding dual face, there is a natural inclusion reversing bijection*

$$\{\check{\tau} \in \mathcal{P}_{\partial\check{\sigma}^{\circ}} | \text{Int}(\check{\tau}) \subseteq \text{Int}(\check{\tau}') \neq \emptyset\} \leftrightarrow p_{\Delta\Delta'}^{-1}(\tau')$$

where the simplex corresponding to $\tau \in p_{\Delta\Delta'}^{-1}(\tau')$ has dimension $d + 1 - \dim \tau$.

Proof. For (1), first note that by Proposition 1.13, $\varphi_{\Delta'}$ is piecewise linear and convex, but not necessarily strictly convex, on the fan $\check{\Sigma}_{\Delta}$. The maximal domains of linearity of $\varphi_{\Delta'}$ define a fan $\check{\Sigma}'$ of not necessarily strictly convex cones in $N_{\mathbb{R}}$, and the fan $\check{\Sigma}_{\Delta'}$ is then obtained by dividing out each cone in $\check{\Sigma}'$ by Δ'^{\perp} . This gives the map of fans $\check{p}_{\Delta\Delta'}$ of (1).

In particular, if $\tau \subseteq \partial\Delta$ and $\check{\tau}$ is the corresponding cone of $\check{\Sigma}_{\Delta}$, $n \in \check{\tau}$, $n \neq 0$, we have that $\langle n, \cdot \rangle = -\varphi_{\Delta}(n)$ is a supporting hyperplane of the face τ . The face of Δ' corresponding to $\check{p}_{\Delta\Delta'}(\check{\tau})$ is then supported by the hyperplane $\langle n, \cdot \rangle = -\varphi_{\Delta'}(n)$. In particular, if $n \in \text{Int}(\check{\tau})$, the image of n in $N_{\mathbb{R}}/\Delta'^{\perp}$ lies in the interior of $\check{p}_{\Delta\Delta'}(\check{\tau})$. In this case, n defines a supporting hyperplane of Δ which intersects Δ only in τ , and defines a supporting hyperplane of Δ'

which intersects Δ' only in the face dual to $\check{p}_{\Delta\Delta'}(\check{\tau})$. This gives a surjective map from the set of faces of Δ to the set of faces of Δ' . Thus to prove (2), we just need to show that this map is the map $p_{\Delta\Delta'}$ described in (2).

To show this, we first need to show that for any $\tau \subseteq \partial\Delta$, there is a unique $\tau' \in \mathcal{P}_*$ such that $\tau' \not\subseteq \partial\Delta$ and $\tau' \cap \partial\Delta = \tau$. This will show bijectivity of $p_{\Delta\Delta'}^1$. Furthermore, we need to show that $\tau' \cap \Delta'$ is the face τ'' of Δ' corresponding to $\check{p}_{\Delta\Delta'}(\check{\tau})$.

To show both these items, let $n \in \text{Int}(\check{\tau})$ be chosen so that $\varphi_K(n) = -1$. Then the affine linear function $-\langle n, \cdot \rangle - \varphi_\Delta(n)$ takes the value 0 on τ and is strictly negative on $\Delta \setminus \tau$, while $-\langle n, \cdot \rangle - \varphi_{\Delta'}(n)$ takes the value 0 on τ'' and is strictly negative on $\Delta' \setminus \tau''$. Since $\varphi_{\Delta'} = \varphi_\Delta + \varphi_K$ by Proposition 1.13, $-\langle n, \cdot \rangle - \varphi_\Delta(n)$ takes the value 0 on τ and the value -1 on τ'' . So

$$\begin{aligned} -\langle n, m \rangle - \varphi_\Delta(n) &= h_*(m) \text{ for } m \in \text{Conv}(\tau, \tau'') =: \tau', \\ -\langle n, m \rangle - \varphi_\Delta(n) &< h_*(m) \text{ for } m \in \Delta \setminus \tau' \end{aligned}$$

by the definition of h_* . Thus $\tau' \in \mathcal{P}_*$ and $\tau' \cap \partial\Delta = \tau$, $\tau' \cap \Delta' = \tau''$, so τ' is as desired.

Finally, given a cell $\tau' \in \mathcal{P}_*$ with $\tau' \cap \partial\Delta = \tau$, $\tau' \not\subseteq \partial\Delta$, we need to show that τ' is as constructed above. Indeed, there is an affine linear function $-\langle n, \cdot \rangle + c$ which coincides with h_* on τ' and is smaller than h_* on $\Delta \setminus \tau'$. Then necessarily the hyperplane $\langle n, \cdot \rangle - c = 0$ intersects Δ precisely in the face τ , so this hyperplane is a support hyperplane for the face τ . Thus $n \in \text{Int}(\check{\tau})$ and $c = -\varphi_\Delta(n)$. Furthermore, for $-\langle n, \cdot \rangle + c$ to take the value -1 on $\tau' \cap \Delta'$, we must have $\varphi_K(n) = -1$. Thus τ' is as constructed in the previous paragraph.

The remaining statements of (2) follow easily from the above discussion.

For (3), by Prop. 1.1, we have

$$\check{\Sigma} = \{\{0\}, \mathbb{R}_{\geq 0}\rho\} \cup \{\check{\tau} | \tau \subseteq \partial\Delta\} \cup \{\mathbb{R}_{\geq 0}\rho + \check{\tau} | \tau \subseteq \partial\Delta\}$$

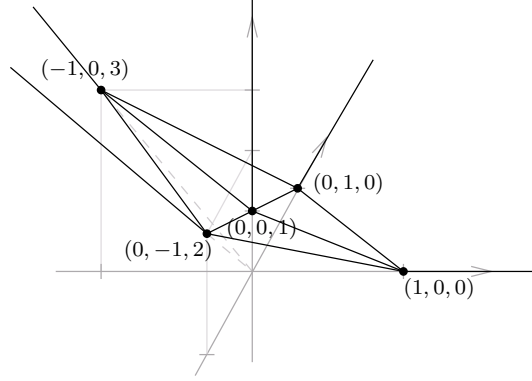
where $\check{\tau} = \text{Cone}(\tau)^\perp \cap \check{\sigma}$. We claim that, for $\omega \in \check{\Sigma}$,

$$\check{\sigma}^\circ \cap \omega = \text{Conv}((\omega \cap \bar{N}) \setminus \{0\}).$$

Indeed, ω is a standard cone which, w.l.o.g, we may assume maximal. Say v_0, v_1, \dots, v_{d+1} are its primitive integral generators with $v_0 = \rho$. Let $v \in \Delta$ be the integral generator of the ray dual to the cone generated by v_1, \dots, v_{d+1} . Then $\langle (p_{\Delta\Delta'}(v), 1), \cdot \rangle = 1$ contains v_0, \dots, v_{d+1} and is a supporting hyperplane of $\check{\sigma}^\circ$. Thus this hyperplane supports $\text{Conv}\{v_0, \dots, v_{d+1}\}$ as a face of $\check{\sigma}^\circ \cap \omega$. Since $\text{Conv}((\omega \cap \bar{N}) \setminus \{0\}) = \{\sum_i \lambda_i v_i | \sum_i \lambda_i \geq 1\}$, the claim follows.

Now $\check{\Sigma}$ induces a subdivision of $\check{\sigma}^\circ$ (resp. $\partial\check{\sigma}^\circ$) which we denote by $\mathcal{P}_{\check{\sigma}^\circ}$ (resp. $\mathcal{P}_{\partial\check{\sigma}^\circ}$), see Fig. 6. At most faces of $\check{\sigma}^\circ$ which contain ρ are effected by the subdivision. By the claim, the cells in $\mathcal{P}_{\partial\check{\sigma}^\circ}$ properly containing ρ are

$$\{\text{Conv}\{\rho, v_1^{\check{\tau}}, \dots, v_{r_{\check{\tau}}}^{\check{\tau}}\} | \tau \subseteq \partial\Delta\}$$

FIGURE 6. $\mathcal{P}_{\check{\sigma}^o}$ for Example 1.6

where $v_1^{\check{\tau}}, \dots, v_{r_{\check{\tau}}}^{\check{\tau}}$ denote the primitive integral generators of $\check{\tau}$, the cone of $\check{\Sigma}$ corresponding to τ . Thus, these cells are in natural bijection with faces of Δ (ρ itself corresponding to Δ). Note that under the duality of Lemma 1.17, (2), the face of $\check{\sigma}^o$ dual to $\tau \subsetneq \Delta$ is an unbounded face. Again by the claim, each such face has one bounded facet, which is

$$\check{\omega}_{\tau} := \text{Conv}\{v_1^{\check{\tau}}, \dots, v_{r_{\check{\tau}}}^{\check{\tau}}\}.$$

By the argument for (2) above, $\check{\omega}_{\tau}$ is dual to $p_{\Delta\Delta'}^1(\tau)$. In turn, the minimal face of $\check{\sigma}^o$ containing both ρ and $\check{\omega}_{\tau}$ is dual to $p_{\Delta\Delta'}(\tau)$. This demonstrates the inclusion reversing bijection. The dimension formula follows from duality and what we said already. \square

The Newton polytope of w is given by

$$\check{\Delta}_0 = \check{\Delta} \cap \check{\sigma}^o = \text{Conv}(\{\rho\} \cup \{(n_{\tau}, \varphi_{\Delta}(n_{\tau})) \mid \tau \subseteq \Delta \text{ a codimension one face of } \Delta\})$$

which is also the convex hull of the bounded faces of $\check{\sigma}^o$. Lemma 1.17 then implies

Lemma 1.20. *We have $\dim \check{\Delta}_0 = d+2$ for $\dim \Delta' > 0$ and $\dim \check{\Delta}_0 = d+1$ for $\dim \Delta' = 0$.*

We set

$$(1.5) \quad W_t = \overline{w^{-1}(t) \cap (\mathbb{C}^*)^{d+2}}$$

where the overline denotes the closure in X_{Σ} . Now, W_t is the strict transform of the hypersurface of X_{σ} given by the same equation because the maps $X_{\Sigma} \rightarrow X_{\Sigma_*} \rightarrow X_{\sigma}$ restricted to $(\mathbb{C}^*)^{d+2}$ give isomorphisms. So we may also take the closure (1.5) in X_{Σ_*} which we then denote by W_t^* . To complete the notation, let \bar{W}_t denote the closure of W_t in $X_{\bar{\Sigma}}$ and \tilde{W}_t the closure in $\tilde{\mathbb{P}}_{\check{\Delta}}$ such that we have a diagram

$$(1.6) \quad \begin{array}{ccccccc} W_t^{\sigma} & \longleftarrow & W_t^* & \longleftarrow & W_t & \hookrightarrow & \bar{W}_t & \xleftarrow{\sim} & \tilde{W}_t \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_{\sigma} & \longleftarrow & X_{\Sigma_*} & \longleftarrow & X_{\Sigma} & \hookrightarrow & X_{\bar{\Sigma}} & \longleftarrow & \tilde{\mathbb{P}}_{\check{\Delta}} \end{array}$$

Given $\tau \in \mathcal{P}$, let $\mathcal{P}_*(\tau)$ denote the smallest cell of \mathcal{P}_* containing τ . For $\tau \in \mathcal{P}_*$, we set

$$\check{\Delta}_\tau = \check{\Delta}_0 \cap \check{\tau},$$

where $\check{\tau}$ denotes the face of $\check{\sigma}^o$ dual to τ .

- Proposition 1.21.** (1) *For $\tau \in \mathcal{P}_*$, the Newton polytope of the hypersurface³ $V(\tau) \cap W_0^*$ in $V(\tau)$ is $\check{\Delta}_\tau$. For $v \in \Delta' \cap M$, the divisor $W_0^* \cap D_v$ is ample in D_v .*
- (2) *The intersection of \bar{W}_t with every closed toric stratum in $X_{\bar{\Sigma}}$ is either empty or smooth for $t = 0$ and $t \in \mathbb{C}$ general. For $\tau \in \mathcal{P}$, the Newton polytope of the hypersurface $V(\tau) \cap W_0$ in $V(\tau)$ is $\check{\Delta}_{\mathcal{P}_*(\tau)}$.*
- (3) *For $t \neq 0$, we have $w^{-1}(t) = W_t$. For $t = 0$, we have*

$$w^{-1}(0) = W_0 \cup \bigcup_{v \in \text{Int}(\Delta) \cap M} D_v,$$

where $D_v \subseteq X_{\Sigma}$ is the toric divisor corresponding to the ray $\text{Cone}(v) \in \Sigma$. Furthermore, $w^{-1}(0)$ is normal crossings.

Proof. Consider the embedding of polytopes $\check{\Delta}_0 \hookrightarrow \check{\sigma}^o$. First assume that $\dim \Delta' > 0$, so that $\dim \check{\Delta}_0 = \dim \check{\sigma}^o$ by Lemma 1.20. In view of Lemma 1.17,(2), for $\tau \in \mathcal{P}_*$, $\text{Cone}(\tau)$ is the normal cone to a face of $\check{\sigma}^o$. From this embedding and the fact that every bounded face of $\check{\sigma}^o$ is also a bounded face of $\check{\Delta}_0$, we see that $\text{Cone}(\tau)$ is contained in a normal cone of $\check{\Delta}_0$, equal to a normal cone of $\check{\Delta}_0$ provided that $\tau \subseteq \Delta'$. Thus the embedding of polytopes induces a morphism of toric varieties $f : X_{\Sigma_*} \rightarrow \mathbb{P}_{\check{\Delta}_0}$. On the other hand, if $\dim \Delta' = 0$, then by Lemma 1.20 one sees that $\check{\Delta}_0$ is a face of $\check{\sigma}^o$, and the projection $\bar{M} \rightarrow \bar{M}/\mathbb{Z}m$ for m the normal vector to the face $\check{\Delta}_0$ induces again a morphism of toric varieties $f : X_{\Sigma_*} \rightarrow \mathbb{P}_{\check{\Delta}_0}$. In either case, $W_0^* = f^{-1}(W_{\check{\Delta}_0})$ for an ample hypersurface $W_{\check{\Delta}_0} \subset \mathbb{P}_{\check{\Delta}_0}$ given by the same equation as W_0^* . Given $\tau \in \mathcal{P}_*$, its dual $\check{\tau}$ is a face of $\check{\sigma}^o$ and we have $V(\tau) = \mathbb{P}_{\check{\tau}}$. The restriction of f to $\mathbb{P}_{\check{\tau}}$ yields the natural map $\mathbb{P}_{\check{\tau}} \rightarrow \mathbb{P}_{\check{\tau} \cap \check{\Delta}_0}$. This is an isomorphism if $\tau \not\subseteq \partial \Delta$. In any case, the Newton polytope of $W_0^* \cap \mathbb{P}_{\check{\tau}}$ is isomorphic to that of $W_{\check{\Delta}_0} \cap \mathbb{P}_{\check{\tau} \cap \check{\Delta}_0}$ which is $\check{\tau} \cap \check{\Delta}_0$ by ampleness. In particular, $W_0^* \cap \mathbb{P}_{\check{\tau}}$ is ample in W_0^* if $\tau \not\subseteq \partial \Delta$.

We have shown (1) and will now deduce (2). The assertion that the Newton polytope of $V(\tau) \cap W_0$ is $\check{\Delta}_{\mathcal{P}_*(\tau)}$ follows from the fact that W_0 is the pullback of W_0^* under the map $X_{\Sigma} \rightarrow X_{\Sigma_*}$ which takes a stratum $V(\tau)$ to $V(\mathcal{P}_*(\tau))$. Since the coefficients of $W_{\check{\Delta}_0}$ are assumed general, $W_{\check{\Delta}_0}$ is $\check{\Delta}_0$ -regular. The remainder of (2) follows from the fact that regularity is preserved under pullback, see [Bat94], Prop. 3.2.1, and the smoothness of $X_{\bar{\Sigma}}$ in a neighbourhood of the closure of W_t .

Finally, for (3), note that, for $t \neq 0$, W_t is the proper transform of the hypersurface W_t^σ in X_σ because W_t^σ is σ -regular, which is not true for W_0^σ because the latter contains

³We use the notation $V(\tau)$ as shorthand for $V(\text{Cone}(\tau))$.

the origin of X_σ . Since $w^{-1}(0)$ is the total transform of W_0^σ , isomorphic over the dense torus, the irreducible components of $w^{-1}(0)$ different from W_0 need to be toric divisors of X_Σ , the set of which is indexed by $\Delta \cap M$. The multiplicities may be computed locally as follows: A standard fact of toric geometry says that the monomial $z^{(n,r)}$ vanishes to order $\langle (n,r), (v,1) \rangle = \langle n, v \rangle + r$ along D_v . In particular, if $v \notin \partial\Delta$, then for any $(n,r) \in \check{\sigma}^\circ$, $\langle (n,r), (v,1) \rangle > 0$, so all the monomials $z^{(n,r)}$ appearing in w vanish on D_v . Furthermore, the monomial z^ρ vanishes to order 1 on D_v , so $D_v \subseteq w^{-1}(0)$ and D_v appears with multiplicity one. On the other hand, if $v \in \partial\Delta$, there is at least one monomial $z^{(n,r)}$ appearing in w not vanishing on D_v . Moreover, all such non-vanishing monomials are linearly independent after restriction to D_v . \square

Corollary 1.22. *For $\tau \in \mathcal{P}$, $\tau \subset \partial\Delta$, denoting by T_τ the torus orbit of X_Σ corresponding to $\text{Cone}(\tau)$, we have*

$$\begin{aligned} w^{-1}(t) \cap T_\tau &\cong H^{\text{codim } \mathcal{P}_*(\tau)-1} \times (\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau)-\dim \tau} & \text{for } t \neq 0, \\ w^{-1}(0) \cap T_\tau &\cong H^{\text{codim } \mathcal{P}_*(\tau)-2} \times (\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau)-\dim \tau+1} \end{aligned}$$

where $\text{codim } \mathcal{P}_*(\tau) = d+1 - \dim \mathcal{P}_*(\tau)$ and H^k denotes a k -dimensional handlebody, i.e., the intersection $H \cap (\mathbb{C}^*)^{k+1}$ for a general hyperplane H in \mathbb{P}^{k+1} .

Proof. Given τ as in the assertion then $\mathcal{P}_*(\tau)$ is a proper face of Δ . By Prop. 1.21 and Lemma 1.19,(3), $\check{\Delta}_{\mathcal{P}_*(\tau)}$, the Newton polytope of $W_0^* \cap T_{\mathcal{P}_*(\tau)}$, is a standard simplex. It is the convex hull of the primitive generators of the face of $\check{\sigma}$ dual to the face $\text{Cone}(\mathcal{P}_*(\tau))$ of σ . Thus the Newton polytope of $W_t^* \cap T_{\mathcal{P}_*(\tau)}$ for $t \neq 0$ is $\text{Conv}(\{0\} \cup \check{\Delta}_{\mathcal{P}_*(\tau)})$. Checking dimensions implies $W_0^* \cap T_{\mathcal{P}_*(\tau)} = H^{d-\dim \mathcal{P}_*(\tau)-1} \times \mathbb{C}^*$ and $W_t^* \cap T_{\mathcal{P}_*(\tau)} = H^{d-\dim \mathcal{P}_*(\tau)}$ for $t \neq 0$. The assertion follows from the fact that the restriction of the map $X_\Sigma \rightarrow X_{\Sigma^*}$ to T_τ is a projection $T_\tau \cong T_{\mathcal{P}_*(\tau)} \times (\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau)-\dim \tau} \rightarrow T_{\mathcal{P}_*(\tau)}$ and $w^{-1}(t) \cap T_\tau$ is the pullback of $W_t^* \cap T_{\mathcal{P}_*(\tau)}$ under this map. \square

Example 1.23. We return to Example 1.5 and Rem. 1.9 where Δ is a reflexive polytope. Then the two Landau-Ginzburg models $w : X_\Sigma \rightarrow \mathbb{C}$ and $\check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$ have similar structure. Both $\check{w}^{-1}(0)$ and $w^{-1}(0)$ have two irreducible components. One of the irreducible components of $\check{w}^{-1}(0)$ is isomorphic to \mathbb{P}_Δ , and one of the irreducible components of $w^{-1}(0)$ is isomorphic to \mathbb{P}_{Δ^*} . The singular locus of $\check{w}^{-1}(0)$ and $w^{-1}(0)$ are Calabi-Yau hypersurfaces in, respectively, \mathbb{P}_Δ and \mathbb{P}_{Δ^*} , and these hypersurfaces are mirror under the Batyrev construction [Bat94].

Example 1.24. We extend Examples 1.6 and 1.12. There are two interior vertices, $v_1 = (1,1)$ and $v_2 = (2,1)$, giving toric components D_1 and D_2 of $w^{-1}(0)$. From the particular choice of triangulation given in Fig. 2, one sees that D_1 and D_2 are given by the fans depicted in Fig. 7 and thus both are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in three points. By the adjunction formula, $D_{v_1} \cap W_0$ and $D_{v_2} \cap W_0$ are rational curves, and one can easily

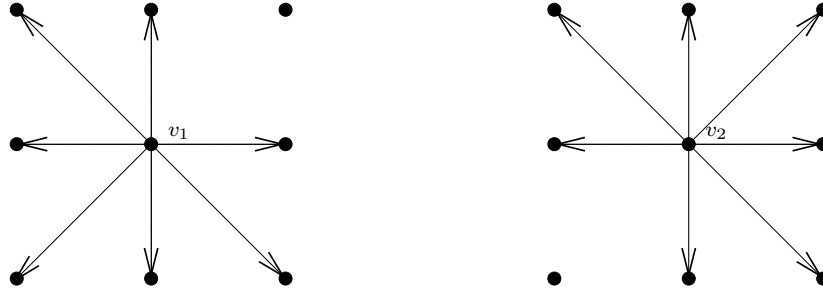
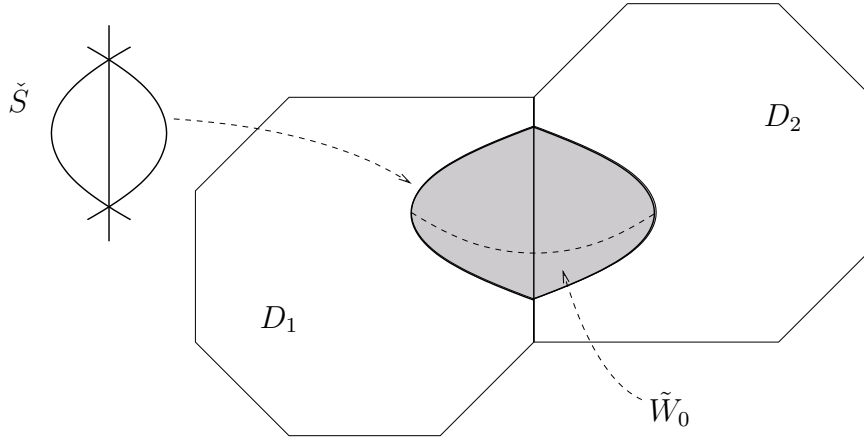
FIGURE 7. The fans of D_1 and D_2 

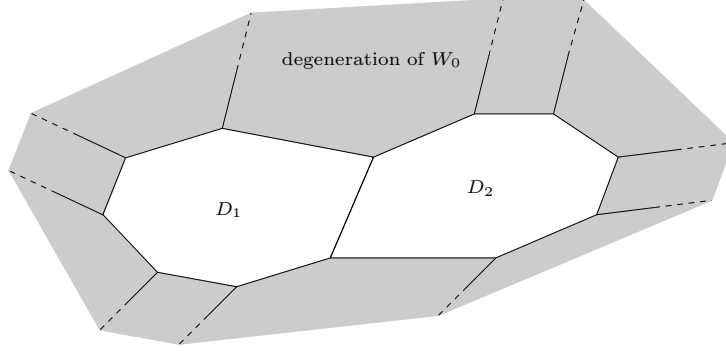
FIGURE 8. The mirror to a genus two curve

compute that $W_0 \cap D_{v_1} \cap D_{v_2}$ consists of two points. We deduce that $w^{-1}(0)$ is as depicted on the right in Fig. 8 and its singular locus \check{S} is a union of three \mathbb{P}^1 's as depicted on the left in Fig. 8. Moreover, \tilde{W}_0 is a rational surface because $\bar{w}^{-1}(0)$ is clearly a type III degeneration of K3 surfaces, being simple normal crossings with triple points, and hence all components must be rational. One can show that \tilde{W}_0 is an Hirzebruch surface \mathbb{F}_2 blown up in 12 points.

Remark 1.25 (The toric degeneration w_t). In Rem. 1.9, we discussed that the family of potentials \tilde{w}_t is only a partial toric degeneration in general. However, on the mirror dual side, if we set

$$w_t = tw + z^\rho,$$

then w_t defines a toric degeneration of $w^{-1}(0)$. Indeed, recall that ρ evaluates to 1 on each primitive generator of a ray in Σ and thus the zero locus of z^ρ is the reduced union of all toric divisors in X_Σ . Since by Prop. 1.21,(3), $w^{-1}(0)$ already contains those corresponding to rays generated by $v \in \Delta'$, we find that w_t degenerates the component \tilde{W}_0 to the union of all toric divisors D_v with $v \in \partial\Delta$. Note, however, that the compactified degeneration of \tilde{W}_0 in $\tilde{\mathbb{P}}_\Delta$ is toric but possibly non-reduced in D_∞ . See Fig. 9 for w_0 in the example of the

FIGURE 9. Toric degeneration of $w^{-1}(0)$ for the genus two curve mirror

genus two curve. Note that the critical locus of w_0 is non-compact for $d \geq 1$ whereas that of w_t for $t \neq 0$ is compact.

The choices for the families w_t and \check{w}_t are motivated by mirror symmetry: up to instanton corrections and up to fixing coefficients, the complex structure parameter t in w_t (resp. \check{w}_t) corresponds to the Kähler parameter given by scaling the sum of the complexified classes of the exceptional divisors in the resolution in X_Σ (resp. $X_{\check{\Sigma}}$) mapping to the origin in X_σ (resp. $X_{\check{\sigma}}$).

1.5. The intersection complex of $w^{-1}(0)$. Recall the following standard definition:

Definition 1.26. Let $X = \bigcup_{i \in I} X_i$ be a strictly normal crossings variety. The *dual intersection complex* Γ_X of X is the simplicial complex with vertices the index set I and there is one simplex $\langle i_0, \dots, i_p \rangle$ for every connected component of $X_{i_0} \cap \dots \cap X_{i_p}$.

Example 1.27. The dual intersection complex of $\check{w}^{-1}(0)$ is a closed interval. In Example 1.5, the dual intersection of $w^{-1}(0)$ is also a closed interval. In Example 1.6, the dual intersection complex of $w^{-1}(0)$ consists of two triangles identified along their boundaries. Hence this complex is homeomorphic to S^2 .

Note that the dual intersection complex of $w^{-1}(0)$ (resp. $\check{w}^{-1}(0)$) is the same as that of $\bar{w}^{-1}(0)$ (resp. $\bar{\check{w}}^{-1}(0)$).

Proposition 1.28. *The set of vertices of the dual intersection complex $\Gamma_{w^{-1}(0)}$ of $w^{-1}(0)$ is*

$$(\Delta' \cap M) \cup \{u\}$$

where u represents W_0 . The precise structure of $\Gamma_{w^{-1}(0)}$ depends on $\dim \Delta'$:

- (1) If $\dim \Delta' \leq d - 1$ then $\Gamma_{w^{-1}(0)}$ is the cone over Δ' . Precisely, the simplices are $\{\langle u \rangle\} \cup \{\langle v_0, \dots, v_p \rangle \mid \text{Conv}\{v_0, \dots, v_p\} \in \mathcal{P}\} \cup \{\langle v_0, \dots, v_p, u \rangle \mid \text{Conv}\{v_0, \dots, v_p\} \in \mathcal{P}\}$.

In particular, $\Gamma_{w^{-1}(0)}$ is topologically a ball of dimension $\dim \Delta' + 1$.

- (2) If $\dim \Delta' = d$ then we have one simplex $\langle u \rangle$, one simplex $\langle v_0, \dots, v_p \rangle$ whenever $\text{Conv}\{v_0, \dots, v_p\} \in \mathcal{P}$, one simplex $\langle v_0, \dots, v_p, u \rangle$ whenever

$$\text{Conv}\{v_0, \dots, v_p\} \in \mathcal{P} \text{ and } \text{Conv}\{v_0, \dots, v_p\} \subseteq \partial \Delta',$$

and two simplices $\langle v_0, \dots, v_p, u \rangle$ whenever

$$\text{Conv}\{v_0, \dots, v_p\} \in \mathcal{P} \text{ and } \text{Conv}\{v_0, \dots, v_p\} \not\subseteq \partial \Delta'.$$

So topologically, $\Gamma_{w^{-1}(0)}$ is obtained by taking two cones over Δ' and gluing them together along the boundary. In particular $\Gamma_{w^{-1}(0)}$ is a $d+1$ -dimensional sphere.

- (3) If $\dim \Delta' = d+1$ then the simplices of $\Gamma_{w^{-1}(0)}$ are

$$\begin{aligned} & \{\langle u \rangle\} \cup \{\langle v_0, \dots, v_p \rangle \mid \text{Conv}\{v_0, \dots, v_p\} \in \mathcal{P}\} \\ & \cup \{\langle v_0, \dots, v_p, u \rangle \mid \text{Conv}\{v_0, \dots, v_p\} \in \mathcal{P} \text{ and } \text{Conv}\{v_0, \dots, v_p\} \subseteq \partial \Delta'\}. \end{aligned}$$

Thus $\Gamma_{w^{-1}(0)}$ is again a $d+1$ -dimensional sphere.

Proof. By Proposition 1.21, the description of the vertices of $\Gamma_{w^{-1}(0)}$ is clear. Let

$$\mathcal{P}_{\Delta'} := \{\omega \in \mathcal{P} \mid \omega \subseteq \Delta'\}.$$

Clearly, for any cell $\omega \in \mathcal{P}_{\Delta'}$ with vertices v_0, \dots, v_p , the toric stratum of X_Σ determined by $\text{Cone}(\omega)$ is just $D_\omega := D_{v_0} \cap \dots \cap D_{v_p}$, hence $\langle v_0, \dots, v_p \rangle$ is a simplex in $\Gamma_{w^{-1}(0)}$. To understand the remaining simplices, we just need to understand $D_\omega \cap W_0$. The family w_t in Remark 1.25 induces a linear equivalence $W_0 \sim \bigcup_{v \in \partial \Delta} D_v$. Restricting this to D_ω yields

$$D_\omega \cap W_0 \sim \bigcup_{v \in \partial \Delta} D_v \cap D_\omega.$$

We are interested in the number of connected components of this divisor class. Using the combinatorial description on the right hand side, this number can be read off from the fan of D_ω . This fan is given by

$$\Sigma(\omega) = \{(\text{Cone}(\tau) + \mathbb{R} \text{Cone}(\omega)) / \mathbb{R} \text{Cone}(\omega) \mid \tau \in \mathcal{P}, \omega \subseteq \tau\}$$

in $\bar{M}_{\mathbb{R}} / (\mathbb{R} \text{Cone}(\omega))$. For two polyhedra $\tau \subseteq \tau' \subseteq \bar{M}_{\mathbb{R}}$ (resp. in $M_{\mathbb{R}}$), we choose any point $x \in \text{Int}(\tau)$ and write

$$T_\tau \tau' = \{c(v - x) \mid c \in \mathbb{R}_{\geq 0}, v \in \tau'\};$$

this is the tangent wedge to τ' along τ . Using this notation, we observe that the rays of $\Sigma(\omega)$ which don't correspond to a divisor $D_v \cap D_\omega$ with $v \in \partial \Delta$ span

$$(T_{\text{Cone}(\omega)} \text{Cone}(\Delta') + \mathbb{R} \text{Cone}(\omega)) / \mathbb{R} \text{Cone}(\omega).$$

By standard toric geometry, the number of connected components of $W_0 \cap D_\omega$ is the same as the number of connected components of

$$(\bar{M}_{\mathbb{R}} / \mathbb{R} \text{Cone}(\omega)) \setminus ((T_{\text{Cone}(\omega)} \text{Cone}(\Delta') + \mathbb{R} \text{Cone}(\omega)) / \mathbb{R} \text{Cone}(\omega)),$$

or equivalently, the number of connected components of $M_{\mathbb{R}} \setminus T_{\omega}\Delta'$.

This now gives the case-by-case description of $\Gamma_{w^{-1}(0)}$. If $\dim \Delta' \leq d-1$, i.e., $\text{codim}(\Delta' \subseteq M_{\mathbb{R}}) \geq 2$, then $M_{\mathbb{R}} \setminus T_{\omega}\Delta'$ is connected and non-empty for all $\omega \in \mathcal{P}_{\Delta'}$, so $\Gamma_{w^{-1}(0)}$ is just a cone over Δ' as described in item (1) of the statement of the Proposition.

If $\dim \Delta' = d$ then, for $\omega \subseteq \partial\Delta'$, $M_{\mathbb{R}} \setminus T_{\omega}\Delta'$ is connected, and there is a unique simplex of $\Gamma_{w^{-1}(0)}$ with vertices u and the vertices of ω . If $\omega \in \mathcal{P}_{\Delta'}, \omega \not\subseteq \partial\Delta'$ then $M_{\mathbb{R}} \setminus T_{\omega}\Delta'$ has two connected components. In this case, there are two simplices with vertices u and the vertices of ω . This gives the description in item (2).

Finally, if $\dim \Delta' = d+1$ then if $\omega \subseteq \partial\Delta'$, there is again a unique simplex of $\Gamma_{w^{-1}(0)}$ with vertices u and the vertices of ω . On the other hand, if $\omega \not\subseteq \partial\Delta'$ then in fact $D_v \cap D_{\omega} = \emptyset$ for all $v \in \partial\Delta$, so D_{ω} is disjoint from W_0 . (Equivalently, $M_{\mathbb{R}} \setminus T_{\omega}\Delta'$ has zero connected components.) \square

2. HOMOLOGICAL MIRROR SYMMETRY AND (CO-)HOMOLOGY

A discussion of the categories related to our construction has already appeared in [KKOY09] and [Ka10]. We just quickly review the main ideas and apply these to the discussion of cohomology. Following [Or11], to a Landau-Ginzburg model (X, w) , we associate the triangulated category $D^b(X, w)$ which is defined as

$$D^b(X, w) = \prod_{t \in \mathbb{A}^1} D_{\text{sing}}^b(w^{-1}(t))$$

where $D_{\text{sing}}^b(w^{-1}(t))$ is the Verdier quotient of $D^b(w^{-1}(t))$, the bounded derived category of coherent sheaves on $w^{-1}(t)$, by $\text{Perf}(w^{-1}(t))$, the full subcategory of perfect complexes (i.e., complexes of locally free sheaves). For a non-critical value t of w , we have $D_{\text{sing}}^b(w^{-1}(t)) = 0$.

The *generalized homological mirror symmetry conjecture* suggests that for mirror dual models (X_{Σ}, w) and $(X_{\tilde{\Sigma}}, \tilde{w})$ given by our construction, there are equivalences of categories

$$(2.1) \quad D^b(X_{\Sigma}, w) \cong \text{DFS}(X_{\tilde{\Sigma}}, \tilde{w})$$

$$(2.2) \quad D^b(X_{\tilde{\Sigma}}, \tilde{w}) \cong \text{DFS}(X_{\Sigma}, w)$$

where $\text{DFS}(X, w)$ is the derived Fukaya-Seidel category of a symplectic fibration $w : X \rightarrow \mathbb{C}$. In general, the Fukaya-Seidel category $\text{FS}(X, w)$ is a conjectural A_{∞} -category at least part of whose objects are Lagrangians which are vanishing cycles over some subsets of the critical locus. It has been rigorously defined for the case where w is a Lefschetz fibration in [Sei01] as follows: Fix a non-critical value λ_0 of w , and choose paths $\gamma_1, \dots, \gamma_n$ in \mathbb{C} which connect the critical values $\lambda_1, \dots, \lambda_n$ of w to λ_0 . Parallel transport of cycles vanishing at λ_i along γ_i should give Lagrangian submanifolds of $w^{-1}(\lambda_0)$. These are the objects of the Fukaya-Seidel category. The morphisms are Floer complexes. Taking twisted complexes and idempotent completion finally yields $\text{DFS}(X, w)$.

2.1. Equivalences for a smooth critical locus: Renormalization flow and Knörrer periodicity. It was pointed out to us by Denis Auroux that, if $S = \text{crit}(\check{w})$ is a smooth compact symplectic manifold, by standard symplectic arguments, $\text{FS}(X_{\check{\Sigma}}, \check{w})$ can be defined. Moreover there is a natural full and faithful functor $\phi : \text{Fuk}(S) \rightarrow \text{FS}(X_{\check{\Sigma}}, \check{w})$ given by mapping a Lagrangian L in S to the set of points in the suitably chosen fixed non-singular fibre which are taken into L under the gradient flow of $\text{Re}(\check{w})$ for some fixed metric. This functor is expected to be essentially surjective when one restricts $(X_{\check{\Sigma}}, \check{w})$ to a neighbourhood of S . In the following discussion, we assume this is the case.

Note that $\text{DFuk}(S)$ is a \mathbb{Z}_2 -graded Calabi-Yau category⁴ and thus its Hochschild homology is \mathbb{Z}_2 -graded and isomorphic to the Hochschild cohomology, see also the next section. By homological mirror symmetry, i.e., by (2.1), $D^b(X_{\Sigma}, w)$ should also be a Calabi-Yau category. Indeed, the anti-canonical divisor of X_{Σ} is trivial, so by [LP11], Thm. 4.1, the subcategory of compact objects is a Calabi-Yau category and it is expected that this generates the entire category.

There is a suggestion of how to refine the grading in [Sei08], §8 for a genus two curve. It is currently unknown whether there is a general way to refine the grading in the cases relevant to us.

For the complex geometry, let $S = \text{crit}(\check{w})$ be given as a complete intersection in a toric variety as in §6.4. By [HW09], Thm. 2 we have an equivalence⁵

$$(2.3) \quad D^b(S) \cong D^b(X, w, \mathbb{Z}^k)$$

where \mathbb{Z}^k indicates a \mathbb{Z}^k -grading given by the $(\mathbb{C}^*)^k$ -action on $\check{w}^{-1}(0)$ induced from the split vector bundle. The hypersurface case is also treated in [Is10], [Sh11]. This equivalence is called *renormalization flow* in the physics literature.

We assumed in §1 that \mathbb{P}_{Δ} is smooth. More generally, one wants to drop the smoothness assumption on \mathbb{P}_{Δ} and work instead with a maximal projective crepant partial resolution $\tilde{\mathbb{P}}_{\Delta}$ of a singular \mathbb{P}_{Δ} as in the Batyrev-Borisov construction [BB94]. Assuming S is a Calabi-Yau manifold after such a resolution, it was shown in [HW09], Thm. 3 that different choices of a resolution give non-canonically equivalent categories $D^b(S)$.

By [Or05], Cor. 3.2, we have

Proposition 2.1 (Orlov). *Let S be smooth and quasi-projective, $f, g \in \Gamma(S, \mathcal{O}_S)$, x a coordinate on \mathbb{A}^1 , $V(g) \subseteq S$ smooth and $f|_{V(g)}$ non-constant then there is a natural equivalence*

$$D^b(V(g), f|_{V(g)}) \cong D^b(S \times \mathbb{A}^1, f + gx).$$

In some sense (2.3) may be viewed as a version of this for the case where $f|_{V(g)} = 0$.

⁴This notion was introduced by Kontsevich and means that this triangulated category supports a right Serre functor which is isomorphic to $[d]$ for some d , where $[\cdot]$ is the shift endo-functor.

⁵The Calabi-Yau assumption in loc.cit. can be dropped for this result.

2.2. Hochschild (co-)homology of a smooth critical locus. On the symplectic side, there are morphisms

$$\mathrm{HH}_{i-d}(\mathrm{Fuk}(S)) \xrightarrow{\alpha} \mathrm{QH}^i(S) \rightarrow \mathrm{HH}^i(\mathrm{Fuk}(S))$$

where the left and right are the Hochschild homology and cohomology of the A_∞ -category $\mathrm{Fuk}(S)$ and the middle one is the quantum cohomology of S . These are conjectured to be isomorphisms under certain conditions, see [Ko94], [AFOO] and for references with SH in place of QH see [Sei07], [Ab10], [Ga]. For the following considerations, let us assume that α is an isomorphism. On the complex side, we have by the Kontsevich-Hochschild-Kostant-Rosenberg theorem for the Hochschild homology and cohomology rings of $D^b(S)$ respectively

$$(2.4) \quad \mathrm{HH}^i(S) = \bigoplus_{p+q=i} H^q(S, \bigwedge^p \mathcal{T}_S),$$

$$(2.5) \quad \mathrm{HH}_i(S) = \bigoplus_{p-q=i} H^q(S, \Omega_S^p).$$

In the classical limit $\mathrm{QH}^i(S)$ becomes $H^i(S)$. Note that when S, \check{S} are smooth Calabi-Yau manifolds⁶, this gives a way of deducing the duality of Hodge numbers $h^{p,q}(S) = h^{d-p,q}(\check{S})$ from the (generalized) homological mirror symmetry conjecture if $d = \dim S \leq 5$. Given all the assumptions, we have

$$(2.6) \quad \begin{aligned} \bigoplus_{p+q=i} H^{p,q}(S) &\cong H^i(S) \\ &\cong \mathrm{QH}^i(S) \\ &\cong \mathrm{HH}_{i-d}(\mathrm{Fuk}(S)) \\ &\cong \mathrm{HH}_{i-d}(D^b(\check{S})) \\ &\cong \bigoplus_{p-q=i-d} H^{p,q}(\check{S}) \\ &\cong \bigoplus_{p+q=2d-i} H^{d-p,q}(\check{S}). \end{aligned}$$

In higher dimensions one needs to add the information of a monodromy action.

2.3. Hochschild (co-)homology of a singular critical locus. We discuss here the case where \check{S} is compact but very singular, e.g., where \check{S} looks like the mirror of a hypersurface S of positive Kodaira dimension. Given a Landau-Ginzburg model $w : X \rightarrow \mathbb{C}$, by [Or11], Thm. 3.5, there is an equivalence of triangulated categories

$$D^b(X, w) \cong \prod_{t \in \mathbb{C}} \mathrm{MF}(X, w - t)$$

where $\mathrm{MF}(W, w)$ is the triangulated category of matrix factorisations defined in loc.cit. It comes with a natural differential $\mathbb{Z}/2\mathbb{Z}$ -graded enhancement $\mathrm{MF}^{\mathrm{dg}}(W, w)$ (see [Or11], Rem 2.6)

⁶Calabi-Yau means for us in particular $h^{0,k}(S) = h^k(S^d)$ for $d = \dim S$.

which is needed in order to define its Hochschild homology and cohomology. By [LP11], 3.1, for $i = 0, 1$, we then have⁷

$$\bigoplus_{k \equiv i \bmod 2} \mathrm{HH}^k(D^d(X, w)) \cong \bigoplus_{k \equiv i \bmod 2} \mathbb{H}^k(X, (\bigwedge^\bullet \mathcal{T}_X, \iota_{dw}))$$

where ι_{dw} denotes contraction by dw . According to [LP11], 3.2 one also expects

$$(2.7) \quad \bigoplus_{k \equiv i \bmod 2} \mathrm{HH}_k(D^b(X, w)) \cong \bigoplus_{k \equiv i \bmod 2} \mathbb{H}^k(X, (\Omega_X^\bullet, dw \wedge)).$$

In fact, one desires a \mathbb{Z} -graded enhancement of $\mathrm{MF}(W, w)$ instead of a $\mathbb{Z}/2\mathbb{Z}$ -graded one in order to be able to “remove” $\oplus_{k \equiv i \bmod 2}$ from the above equalities. However, note “removing” cannot hold literally. For example, for the setup of (2.3), (2.7) becomes (2.5). The right hand side of (2.5) involves individual Hodge groups. On the other hand, in the cohomology of the sheaf of vanishing cycles, appearing on the right hand side of (2.7), we can’t identify this splitting. Assuming the generalized homological mirror symmetry conjecture holds for a mirror pair $(X_{\check{S}}, \check{w}), (X_\Sigma, w)$ of our construction in §1 (i.e., with $S = \mathrm{crit}(\check{w})$ smooth) we deduce for $i = 0, 1$,

$$(2.8) \quad \bigoplus_{k \equiv i \bmod 2} H^k(S, \mathbb{C}) \cong \bigoplus_{k \equiv i \bmod 2} \mathbb{H}^{k-d}(\check{S}, \mathcal{F}_{\check{S}})$$

by using (2.6) on one side of the mirror pair (unlike in the Calabi-Yau case where it applies on both sides) and combining it with the functor ϕ from the beginning of §2.1, with (2.7) and Thm 0.1. In fact, we prove a much stronger result in Thm. 0.2. This suggests there might be a more refined version of homological mirror symmetry in this situation.

2.4. A conjecture on the Hochschild cohomology. In the light of the previous discussion, the calculations of this paper really only represent half of what one should expect, in the following sense. Except for the Calabi-Yau case, we expect that the Hochschild homology and cohomology of $D^b(S)$ differ; e.g., the (co-)homologies generally differ in the Fano case. However, (2.8) only deals with the Hochschild homology. We also wish to identify the relevant Hochschild cohomology group of $D^b(S)$ on the singular mirror \check{S} . Since $\mathrm{DFuk}(S)$ is a Calabi-Yau category and therefore Hochschild cohomology is isomorphic to Hochschild homology, this cannot be understood from (2.1), but we should look at (2.2) instead. However, as mentioned before, there is no known construction of $\mathrm{FS}(X_\Sigma, w)$ or a conjecturally equivalent category $\mathrm{Fuk}(\check{S}, \mathcal{F}_{\check{S}})$. Furthermore, we don’t know how to relate this to a version of quantum cohomology for $(\check{S}, \mathcal{F}_{\check{S}})$. Despite this being all rather speculative, we still have some evidence that the cohomologies match up. In fact, we conjecture

⁷As mentioned in the introduction of loc.cit., the requirement of a single critical value 0 as assumed in loc.cit. can easily be removed in order to get the result stated here.

Conjecture 2.2. *For S, \check{S} the singular loci of $\check{w}^{-1}(0), w^{-1}(0)$ for a mirror pair of Landau-Ginzburg models as constructed in §1, we have*

$$\mathrm{HH}^i(S) \cong H^i(\check{S}, \mathbb{C}).$$

As evidence for this conjecture, note this holds when S is a curve. Indeed, we have, for $g \geq 2$ the genus of S ,

$$\begin{aligned} \mathrm{HH}^0(S) &= H^0(S, \mathcal{O}_S) \cong \mathbb{C} \\ \mathrm{HH}^1(S) &= H^1(S, \mathcal{O}_S) \oplus H^0(S, \mathcal{T}_S) \cong \mathbb{C}^g \\ \mathrm{HH}^2(S) &= H^1(S, \mathcal{T}_S) \cong \mathbb{C}^{3g-3}. \end{aligned}$$

Assuming that S is defined as a hyperplane section of \mathbb{P}_Δ for Δ satisfying Assumption 1.4, then it is standard that g is $\# \mathrm{Int}(\Delta) \cap M = \# \Delta' \cap M$. Now \check{S} is connected, so $H^0(\check{S}, \mathbb{C}) = \mathbb{C} \cong \mathrm{HH}^0(S)$. The curve \check{S} is a union of rational components, but it is easy to see that its intersection complex is a graph of genus g , and thus $H^1(\check{S}, \mathbb{C}) \cong \mathbb{C}^g \cong \mathrm{HH}^1(S)$.

Finally,

$$H^2(\check{S}, \mathbb{C}) \cong \mathbb{C}^{\# \text{ of irreducible components of } \check{S}}.$$

From the combinatorial description of \check{S} from Prop. 1.28, one sees that this number of irreducible components is $e + b$, where e is the number of edges e of $\mathcal{P} \cap \Delta'$ and $b := \# \partial \Delta' \cap M$. Note that b is also the number of edges of $\mathcal{P} \cap \Delta'$ contained in $\partial \Delta'$. Let f be the number of two-dimensional cells (standard simplices) in $\mathcal{P} \cap \Delta'$. Then the area A of Δ' is $f/2$, but by Pick's Theorem we also have $A = i + b/2 - 1$, where $i = \# \mathrm{Int}(\Delta') \cap M$. Also $1 = \chi(\Delta') = (b + i) - e + f$, where χ denotes the topological Euler characteristic (using compactly supported cohomology). From these two equations one calculates that $e + b = 3(i + b) - 3 = 3g - 3$, as desired.

We have also checked that Conjecture 2.2 holds when S is a quintic surface in \mathbb{P}^3 .

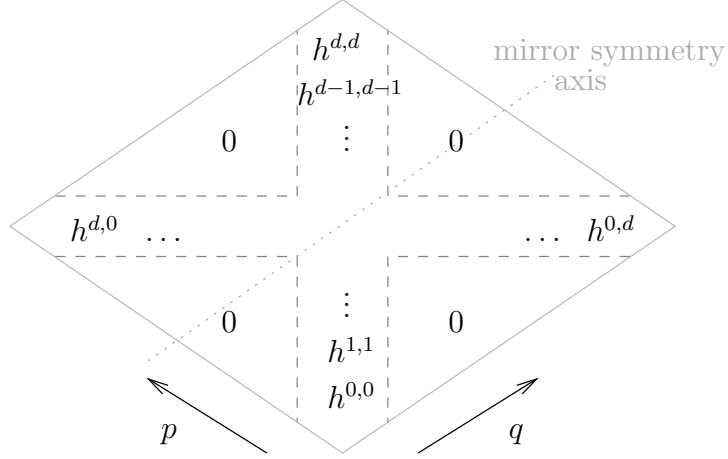
3. HODGE NUMBERS OF HYPERSURFACES IN PROJECTIVE TORIC VARIETIES

In this section, we recall the results of Danilov and Khovanskii about the Hodge numbers of a regular hypersurface in a non-singular toric variety. We will later compare this with the Hodge numbers of the mirror of such a hypersurface.

We recall:

Definition 3.1. For a variety X , one defines the (p, q) -th and p -th *Hodge-Deligne numbers*

$$\begin{aligned} e^{p,q}(X) &= \sum_i (-1)^i h^{p,q} H_c^i(X, \mathbb{C}), \\ e^p(X) &= \sum_q e^{p,q}(X) \stackrel{q=q'+k}{\stackrel{i=p+q'}{=}} (-1)^p \sum_{q',k} (-1)^{q'} h^{p,q'+k} H_c^{p+q'}(X, \mathbb{C}). \end{aligned}$$

FIGURE 10. The Hodge diamond of S

We fix a polytope $\Delta \subseteq M_{\mathbb{R}}$ as usual with $\dim \Delta = \dim M_{\mathbb{R}} = d + 1$ and assume that it comes with a polyhedral decomposition \mathcal{P} into standard simplices. We also assume that \mathbb{P}_{Δ} is a non-singular toric variety. Note that \mathbb{P}_{Δ} comes with the ample line bundle $\mathcal{O}_{\mathbb{P}_{\Delta}}(1)$. We pick a general section of this line bundle, defining a non-singular hypersurface S in \mathbb{P}_{Δ} .

Proposition 3.2. (1) $h^{p,q}(S) = 0$ unless $p = q$ or $p + q = d$.
 (2) For $\tau \in \mathcal{P}$, let $\Delta(\tau)$ be the minimal face of Δ containing τ . Then

$$\begin{aligned} (-1)^p e^p(S) = \sum_q (-1)^q h^{p,q}(S) &= - \sum_{\tau \subseteq \Delta} (-1)^{\dim \tau} \binom{\dim \tau}{p+1} \\ &\quad + \sum_{\tau \in \mathcal{P}} (-1)^{\dim \tau} \binom{\dim \Delta(\tau) - \dim \tau}{p+1} \end{aligned}$$

(3) For $2p > d$,

$$h^{p,p}(S) = h^{p+1,p+1}(\mathbb{P}_{\Delta}) = (-1)^{p+1} \sum_{\tau \subseteq \Delta} (-1)^{\dim \tau} \binom{\dim \tau}{p+1}$$

and

$$h^{p,d-p}(S) = \sum_{\tau \in \mathcal{P}} (-1)^{d-p+\dim \tau} \binom{\dim \Delta(\tau) - \dim \tau}{p+1}.$$

Proof. This is just rewriting formulas of [DK86], 5.5. We begin with

$$\sum_q (-1)^{p+q} h^{p,q}(S) = (-1)^{p+1} \sum_{\tau \subseteq \Delta} (-1)^{\dim \tau} \binom{\dim \tau}{p+1} - \sum_{\omega \subseteq \Delta} (-1)^{\dim \omega} \varphi_{\dim \omega - p}(\omega).$$

Here the sum is over all faces τ (resp. ω) of Δ , and

$$\varphi_i(\omega) = (-1)^i \sum_{j \geq 1} (-1)^j \binom{\dim \omega + 1}{i - j} l^*(j\omega)$$

with $l^*(j\omega)$ the number of interior integral points in $j\omega$. Using \mathcal{P} , we can compute this as follows.

If τ is a standard i -dimensional simplex, then $l^*(j\tau) = \binom{j-1}{i}$. Thus, if ω is a face of Δ , we have

$$l^*(j\omega) = \sum_{\substack{\tau \in \mathcal{P} \\ \tau \subseteq \omega, \tau \not\subseteq \partial\omega}} l^*(j\tau) = \sum_{\substack{\tau \in \mathcal{P} \\ \tau \subseteq \omega, \tau \not\subseteq \partial\omega}} \binom{j-1}{\dim \tau}.$$

We insert this in the above expression for $\varphi_i(\omega)$ and apply Prop. A.1,(1) to get

$$\varphi_i(\omega) = \sum_{\substack{\tau \in \mathcal{P} \\ \tau \subseteq \omega, \tau \not\subseteq \partial\omega}} (-1)^{i+\dim \tau+1} \binom{\dim \omega - \dim \tau}{\dim \omega + 1 - i},$$

and we conclude (2). (1) follows from the Lefschetz theorem proved in 3.7 of [DK86], and the formula for $h^{p,p}$ in (3) follows from that Lefschetz theorem and [DK86], 2.5. The formula for $h^{p,d-p}(S)$ then comes from (2) and the fact that $(-1)^p e^p(S) = (-1)^p h^{p,p}(S) + (-1)^{d-p} h^{p,d-p}(S)$. \square

The statements of [DK86], 1.6 and 1.8 give:

Theorem 3.3. *For $X = \sqcup_i X_i$ a disjoint union and X, Y, X_i varieties, we have*

- (1) $e^{p,q}(X) = \sum_i e^{p,q}(X_i)$, in particular $e^p(X) = \sum_i e^p(X_i)$,
- (2) $e^{p,q}(X \times Y) = \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} e^{p_1,q_1}(X) e^{p_2,q_2}(Y)$, in particular $e^p(X \times Y) = \sum_k e^{p-k}(X) e^k(Y)$.

We give a proof of a lemma that we will need later:

Lemma 3.4. *Recall a handlebody H^k is the intersection of a general hyperplane in \mathbb{P}^{k+1} with $(\mathbb{C}^*)^{k+1}$. We have $e^{p,q}(H^k \times (\mathbb{C}^*)^l) = 0$ for $p \neq q$ and*

$$e^{p,p}(H^k \times (\mathbb{C}^*)^l) = (-1)^{p+k+l} \left(\binom{k+l+1}{p+1} - \binom{l}{p+1} \right).$$

Proof. By [DK86], 1.10, $e^{p,q}((\mathbb{C}^*)^l)$ is zero for $p \neq q$ and $e^{p,p}((\mathbb{C}^*)^l) = (-1)^{p+l} \binom{l}{p}$. Note that if H denotes a hyperplane in \mathbb{P}^{k+1} then we have the motivic sum $H = \bigsqcup_{i=0}^k \binom{k+2}{i+2} H^i$.

Since $H \cong \mathbb{P}^k$, by induction over k using Prop. A.1,(1), we get $e^{p,q}(H^k) = 0$ for $p \neq q$ and $e^{p,p}(H^k) = (-1)^{p+k} \binom{k+1}{p+1}$. The product formula Thm 3.3,(2) yields

$$e^p(H^k \times (\mathbb{C}^*)^l) = \sum_{p_1 \geq 0} (-1)^{p_1+k} (-1)^{p-p_1+l} \binom{k+1}{p_1+1} \binom{l}{p-p_1}$$

and the assertion follows from Prop. A.1,(2). \square

4. THE MIXED HODGE STRUCTURE ON THE COHOMOLOGY OF THE VANISHING CYCLES

We review the notion of the sheaf of vanishing cycles from [Del73] and the Hodge structure on its cohomology as given in [St75], [PS08].

4.1. Vanishing cycles of a semistable degeneration. We fix a proper map $f : \bar{X} \rightarrow O$, where O is the unit disk and f is smooth away from $f^{-1}(0)$. Consider the following diagram:

$$\begin{array}{ccccccc} Y & \xrightarrow{i} & \bar{X} & \xleftarrow{k} & \tilde{X}^* & \longrightarrow & \tilde{O}^* \\ \downarrow & & \downarrow f & \swarrow j^Y & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & O & \longleftarrow & \bar{X}^* & \longrightarrow & O^* \end{array}$$

Here Y is the fibre over $0 \in O$, i the inclusion, $\bar{X}^* = \bar{X} \setminus Y$, $O^* = O \setminus \{0\}$, \tilde{O}^* the universal cover of O^* and $\tilde{X}^* = \bar{X}^* \times_{O^*} \tilde{O}^*$ the pullback of the family $\bar{X}^* \rightarrow O^*$ to \tilde{O}^* . The map j^Y is the inclusion and the map k the projection $\tilde{X}^* \rightarrow \bar{X}^*$ followed by j^Y .

Definition 4.1. The functor $\psi_f : D^+(\bar{X}, \mathbb{Z}) \rightarrow D^+(Y, \mathbb{Z})$ from the derived category of sheaves of abelian groups on \bar{X} to the derived category of sheaves of abelian groups on Y is defined by, for $\mathcal{F} \in D^+(\bar{X}, \mathbb{Z})$,

$$\psi_f(\mathcal{F}) = i^{-1} \mathbf{R}k_*(k^{-1}(\mathcal{F})).$$

This is the *sheaf of nearby cycles* of \mathcal{F} . There is a natural map

$$\text{sp} : i^{-1} \mathcal{F} \rightarrow \psi_f(\mathcal{F}).$$

The cone of this map in $D^+(Y, \mathbb{Z})$ is $\phi_f(\mathcal{F})$, the *sheaf of vanishing cycles* of \mathcal{F} .

For a complex of sheaves \mathcal{F} , we denote by $\mathcal{H}^i(\mathcal{F})$ the i -th cohomology sheaf of the complex, and put⁸

$$R^i \psi_f(\mathcal{F}) := \mathcal{H}^i(\psi_f(\mathcal{F})),$$

$$R^i \phi_f(\mathcal{F}) := \mathcal{H}^i(\phi_f(\mathcal{F})).$$

If $g : \bar{X} \rightarrow C$ is a proper map to a Riemann surface C and $p \in C$, we denote by $\psi_{g,p}$ and $\phi_{g,p}$ the above functors on the category of complexes of sheaves on $g^{-1}(O)$ for a disk

⁸Note that \mathcal{H}^i commutes with i^{-1} since \bar{X} retracts to Y .

O centered at p , small enough so that p is the only critical value of g in O . Clearly, the image of the functor is independent of the size of the disk. This now explains the notation of Theorem 0.1.

Theorem 4.2. *Let $f : \bar{X} \rightarrow O$ be a proper morphism over a disk O , and suppose $X \subseteq \bar{X}$ is an open subset such that, with $D := \bar{X} \setminus X$ flat over O , $Y = f^{-1}(0)$, $D \cup Y$ is a reduced normal crossings divisor. Let $j^D : X \rightarrow \bar{X}$ be the inclusion and*

$$Y = \bigcup_{i=1}^{N_Y} Y_i \quad \text{and} \quad D = \bigcup_{i=1}^{N_D} D_i$$

be the decomposition into irreducible components. We define the sheaf on \bar{X}

$$\mathbb{C}_{Y^1} := \bigoplus_{i=1}^{N_Y} \mathbb{C}_{Y_i}$$

where \mathbb{C}_{Y_i} denotes the (push-forward of) the constant sheaf on Y_i with coefficients in \mathbb{C} . We define \mathbb{C}_{D_i} and \mathbb{C}_{D^1} similarly. We set

$$\mathbb{C}'_{Y^1 \cup D^1} := \text{coker} \left(\mathbb{C}_Y \xrightarrow{(\text{Diag}, 0)} \mathbb{C}_{Y^1} \oplus \mathbb{C}_{D^1} \right)$$

where Diag is the linear map sending 1 to ρ with $\rho_i = 1$ for $1 \leq i \leq N_Y$. Then

$$(1) \quad R^q \psi_f(\mathbf{R}j_*^D \mathbb{C}_X) = \bigwedge^q \mathbb{C}'_{Y^1 \cup D^1}.$$

Under the additional assumption of

$$(4.1) \quad \text{Sing}(Y) \cap D = \emptyset,$$

we have

$$(2) \quad R^q \phi_f(\mathbf{R}j_*^D \mathbb{C}_X) \text{ is supported on } \text{Sing}(Y) \text{ for } q \geq 0,$$

$$(3) \quad R^q \phi_f(\mathbf{R}j_*^D \mathbb{C}_X) = \begin{cases} 0 & \text{if } q = 0; \\ \bigwedge^q \mathbb{C}'_{Y^1 \cup D^1}|_{Y \setminus D} & \text{if } q > 0, \end{cases}$$

$$(4) \quad R^q \phi_f(\mathbf{R}j_*^D \mathbb{C}_X) = R^q \phi_f(\mathbb{C}_{\bar{X}}) \quad \text{for } q \geq 0.$$

Proof. For $U \subseteq \bar{X}$ a small neighbourhood of a point p in Y , since $Y \cup D$ is normal crossings, $U \setminus (Y \cup D)$ has the homotopy type of $(S^1)^{n_Y} \times (S^1)^{n_D}$ where n_Y, n_D are the numbers of irreducible components of Y , resp. D , passing through p . We use the Eilenberg-Moore spectral sequence to translate the Cartesian square

$$\begin{array}{ccc} U & \xleftarrow{k} & k^{-1}(U) \\ f \downarrow & & \downarrow \\ O & \xleftarrow{\quad} & \tilde{O}^* \end{array}$$

into cohomology. It degenerates to the generalized Künneth formula

$$H^\bullet(k^{-1}(U), \mathbb{Z}) = H^\bullet((S^1)^{n_Y} \times (S^1)^{n_D}, \mathbb{Z}) \otimes_{H^\bullet(S^1, \mathbb{Z})} H^\bullet(\mathbb{R}, \mathbb{Z})$$

where the map $H^\bullet(S^1, \mathbb{Z}) = \mathbb{Z}[x]/x^2 \rightarrow H^\bullet(\mathbb{R}, \mathbb{Z}) = \mathbb{Z}$ is given by sending $x \mapsto 0$ and $H^\bullet(S^1, \mathbb{Z}) \rightarrow H^\bullet((S^1)^{n_Y} \times (S^1)^{n_D}, \mathbb{Z}) = (\mathbb{Z}[y]/y^2)^{\otimes n_Y} \otimes (\mathbb{Z}[d]/d^2)^{\otimes n_D}$ is given by $x \mapsto \rho = \sum_{i=1}^{n_Y} 1^{\otimes(i-1)} \otimes y \otimes 1^{\otimes n_Y + n_D - i}$. Rewriting yields

$$H^q(k^{-1}(U), \mathbb{C}) = H^q(U, \mathbb{C})/(\rho) = \Gamma(U, \bigwedge^q \mathbb{C}'_{Y^1 \cup D^1})$$

and proves (1). Part (2) follows from the fact that the adjunction $i^{-1} \mathbf{R}j_*^D \mathbb{C}_X \rightarrow \psi_f(\mathbf{R}j_*^D \mathbb{C}_X)$ is a quasi-isomorphism outside of $\text{Sing}(Y)$ which can be seen from (1). Part (3)-(4) follows from the fact that $\mathbb{C}_{\bar{X}} \rightarrow \mathbf{R}j_*^D \mathbb{C}_X$ is a quasi-isomorphism away from D . \square

Example 4.3. Applying this to the case of $\bar{w} : \bar{w}^{-1}(O) \rightarrow O$, we take $D = \tilde{X}_{\bar{\Sigma}} \setminus X_{\bar{\Sigma}}$. We note that $\mathbb{C}'_{Y^1 \cup D^1}|_{\bar{w}^{-1}(0) \cap X_{\bar{\Sigma}}}$ is \mathbb{C}_S , where $S = D_0 \cap \tilde{W}_0$, in the notation of Proposition 1.8, is a hypersurface in \mathbb{P}_Δ . Thus

$$R^q \phi_{\bar{w}}(\mathbf{R}j_*^D \mathbb{C}_X) = \begin{cases} 0 & \text{if } q \neq 1; \\ \mathbb{C}_S & \text{if } q = 1. \end{cases}$$

From this we conclude that

$$(4.2) \quad \mathbb{H}^q(\bar{w}^{-1}(0), \phi_{\bar{w}}(\mathbf{R}j_*^D \mathbb{C}_X)) = H^{q-1}(S, \mathbb{C}).$$

Most useful for the next sections is Thm. 4.2,(4) because it enables us to work with the vanishing cycles of a compact degeneration which involves slightly less technology.

4.2. Mixed Hodge structure. Our goal in this section is to define a mixed Hodge structure on the hypercohomology groups of $\phi_f \mathbb{C}_{\bar{X}}$. To do so, we shall identify a cohomological mixed Hodge complex whose \mathbb{C} -part is quasi-isomorphic to $\phi_f \mathbb{C}_{\bar{X}}$.

The notion of a cohomological mixed Hodge complex is due to Deligne [DelTH], III. We will always ignore the \mathbb{Z} -module structure of these complexes, and will only be concerned with \mathbb{Q} -module structures. Moreover, we restrict ourselves to normalized ones in the sense of [PS08], Rem. 3.15, i.e., with an explicit comparison pseudo-morphism β given as

$$(4.3) \quad (\mathcal{K}_{\mathbb{Q}}^\bullet, W) \xrightarrow{\beta_1} (\mathcal{K}_{\mathbb{C}}^\bullet, W) \xleftarrow{\beta_2} (\mathcal{K}_{\mathbb{C}}^\bullet, W, F)$$

where β_2 is a filtered quasi-isomorphism and β_1 become such after tensoring with \mathbb{C} . A map of cohomological mixed Hodge complexes is a map on all three terms compatible with the β_i .

Recall that to a filtered complex of sheaves K^\bullet on a topological space with increasing filtration W one associates a spectral sequence $E_\bullet(K^\bullet, W)$ with

$$(4.4) \quad E_1^{p,q}(K^\bullet, W) = \mathbb{H}^{p+q}(\text{Gr}_{-p}^W K^\bullet) \Rightarrow \mathbb{H}^{p+q}(K^\bullet).$$

To apply this to a complex with decreasing filtration F^\bullet , one sets $F_n = F^{-n}$.

Lemma 4.4. *Let $\phi : \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$ given by*

$$\begin{array}{ccccc} (\mathcal{K}_{\mathbb{Q}}^\bullet, W) & \xrightarrow{\beta_1} & (\mathcal{K}_{\mathbb{C}}^\bullet, W) & \xleftarrow{\beta_2} & (\mathcal{K}_{\mathbb{C}}^\bullet, W, F) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{L}_{\mathbb{Q}}^\bullet, W) & \xrightarrow{\beta_1} & (\mathcal{L}_{\mathbb{C}}^\bullet, W) & \xleftarrow{\beta_2} & (\mathcal{L}_{\mathbb{C}}^\bullet, W, F) \end{array}$$

be a map of cohomological mixed Hodge complexes which induces

- (1) *an injection (resp. surjection) upon applying $\mathrm{Gr}_F \mathrm{Gr}^W$,*
- (2) *an injection (resp. surjection) upon applying $E_1(\cdot, W)$*

then $\mathrm{coker}(\phi)$ (resp. $\mathrm{ker}(\phi)$) naturally is a cohomological mixed Hodge complex.

Proof. We only prove the statement for $\mathrm{coker}(\phi)$ since $\mathrm{ker}(\phi)$ works analogously. We first show the degeneration of $E_\bullet(\mathrm{Gr}^W \mathcal{L}/\mathcal{K}, F)$ at E_1 for which we only need assumption (1). We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbb{H}^k(\mathrm{Gr}_F \mathrm{Gr}^W \mathcal{L}^\bullet) & \longrightarrow & \mathbb{H}^k(\mathrm{Gr}_F \mathrm{Gr}^W (\mathcal{L}/\mathcal{K})^\bullet) & \longrightarrow & \mathbb{H}^{k+1}(\mathrm{Gr}_F \mathrm{Gr}^W \mathcal{K}^\bullet) & & \\ \parallel & & \uparrow & & \parallel & & \\ 0 \longrightarrow \mathbb{H}^k(\mathrm{Gr}_F \mathrm{Gr}^W \mathcal{L}^\bullet) & \longrightarrow & \mathbb{H}^k(\mathrm{Gr}_F \mathrm{Gr}^W \mathrm{Cone}_M(\phi)^\bullet) & \longrightarrow & \mathbb{H}^{k+1}(\mathrm{Gr}_F \mathrm{Gr}^W \mathcal{K}^\bullet) & \longrightarrow & 0 \end{array}$$

where $\mathrm{Cone}_M(\phi)$ denotes the mixed cone of ϕ as in [PS08], Thm 3.22 and the middle vertical map is induced by $\mathcal{K}^\bullet[1] \oplus \mathcal{L}^\bullet \rightarrow (\mathcal{L}/\mathcal{K})^\bullet$ which is the natural map factoring through the projection to \mathcal{L}^\bullet . The zeros on the left and right of the bottom row arise because the sequence defining $\mathrm{Cone}_M(\phi)^\bullet$ as an extension of \mathcal{L}^\bullet and $\mathcal{K}^\bullet[1]$ splits after applying Gr_W . A diagram chase shows the surjectivity of the middle vertical map. Since this map comes from a map of filtered complexes, it is part of a map of spectral sequences $E_\bullet(\mathrm{Gr}^W \mathrm{Cone}_M(\phi)^\bullet, F) \rightarrow E_\bullet(\mathrm{Gr}^W (\mathcal{L}/\mathcal{K})^\bullet, F)$. The surjectivity at E_1 and the degeneration of the first at E_1 implies the degeneration of the second at E_1 .

The first row in the above diagram extends to a long exact sequence where we may interchange \mathbb{H}^\bullet and Gr_F in each term. From this and [DelTH], II, Prop. (1.1.11), (i), it follows that the maps in the long exact sequence

$$\cdots \rightarrow \mathbb{H}^k(\mathrm{Gr}^W \mathcal{K}^\bullet) \rightarrow \mathbb{H}^k(\mathrm{Gr}^W \mathcal{L}^\bullet) \rightarrow \mathbb{H}^k(\mathrm{Gr}^W (\mathcal{L}/\mathcal{K})^\bullet) \rightarrow \cdots$$

are strict with respect to F . By assumption (2), the connecting homomorphisms are all trivial and this long exact sequence splits in short exact sequences. The strictness implies that we have an isomorphism of filtered vector spaces $(\mathbb{H}^k(\mathrm{Gr}_l^W \mathcal{L}^\bullet)/\mathrm{im}(\phi), F) \rightarrow (\mathbb{H}^k(\mathrm{Gr}_l^W (\mathcal{L}/\mathcal{K})^\bullet), F)$. Since the first is a Hodge structure of weight $k+l$, so is the second. \square

We assume the setup and notation of §4.1 as given in Thm 4.2. In addition, we denote by Y^k the normalization of

$$\coprod_{i_1 < \dots < i_k} Y_{i_1} \cap \dots \cap Y_{i_k}$$

and by a_i^Y the projection $Y^i \rightarrow \bar{X}$. We are going to recall the construction of the mixed Hodge structure on the hypercohomology of $\phi_f(\mathbb{C}_{\bar{X}})$ following [St75], [PS08]. This is done by giving a map of cohomological mixed Hodge complexes resolving $i^{-1}\mathbb{C}_{\bar{X}} \rightarrow \psi_f \mathbb{C}_{\bar{X}}$. Taking the mixed cone, we will then obtain a cohomological mixed Hodge complex resolving

$$\phi_f \mathbb{C}_{\bar{X}} = \text{Cone}(\mathbb{C}_{\bar{X}}|_Y \rightarrow \psi_f \mathbb{C}_{\bar{X}}).$$

We have the increasing filtrations W^Y defined on $\Omega_{\bar{X}}^\bullet(\log Y)$ by

$$W_k^Y \Omega_{\bar{X}}^p(\log Y) = \Omega_{\bar{X}}^k(\log Y) \wedge \Omega_{\bar{X}}^{p-k}.$$

Moreover, there is the Hodge filtration

$$F^k \Omega_{\bar{X}}^\bullet(\log Y) = \Omega_{\bar{X}}^{\bullet \geq k}(\log Y).$$

Consider the double complex⁹

$$A^{p,q} = \Omega_{\bar{X}}^{p+q+1}(\log Y) / W_q^Y \Omega_{\bar{X}}^{p+q+1}(\log Y).$$

The first differential is the exterior derivation and the second differential is given by wedging with $d \log f = f^* d \log t$, i.e., we fix a coordinate t of O . For a double complex $C^{\bullet, \bullet}$, we denote the total complex by C^\bullet . We have three filtrations on A^\bullet given by the rules

$$(4.5) \quad W_k A^r = \bigoplus_{p+q=r} W_{2q+k+1}^Y \Omega_{\bar{X}}^{p+q+1}(\log Y) / W_q^Y \Omega_{\bar{X}}^{p+q+1}(\log Y)$$

and respectively in terms of the filtrations on A^r and $\Omega_{\bar{X}}^{p+q+1}(\log Y) / W_q^Y \Omega_{\bar{X}}^{p+q+1}(\log Y)$

$$(4.6) \quad W_k^Y = \bigoplus_{p+q=r} W_{k+q+1}^Y \quad F^k = \bigoplus_{p+q=r} F^{k+q+1}.$$

We have $F^k A^{\bullet, \bullet} = A^{\bullet \geq k, \bullet}$. The injection $d \log f \wedge : \Omega_{\bar{X}/O}^p(\log Y) \otimes \mathcal{O}_Y \rightarrow A^{p,0}$ turns $A^{\bullet, \bullet}$ into a resolution of $\Omega_{\bar{X}/O}^\bullet(\log Y) \otimes \mathcal{O}_Y$.

By [St75], Thm 4.19, A^\bullet is the \mathbb{C} -part of a cohomological mixed Hodge complex. There is an endomorphism of this double complex $\nu : A^{p,q} \rightarrow A^{p-1,q+1}$ simply given by the natural projection modulo W_{q+1}^Y . We have $\log T = 2\pi i \nu$ where T is the monodromy transform on cohomology, see [PS08], Thm 11.21 and Cor. 11.17. We have $\ker(\nu)^\bullet = W_0^Y A^\bullet$ with the filtrations W and F induced from A^\bullet . The injection

$$\text{sp} : \ker(\nu)^\bullet \rightarrow A^\bullet$$

⁹Note that we adapt to the original notation by Steenbrink [St75]. The two indices p, q are swapped in [PS08].

is bifiltered. By [PS08], §11.3.1, $\ker(\nu)^\bullet$ is a cohomological mixed Hodge complex computing $H^\bullet(Y, \mathbb{C})$. A useful description for the rational structure for A^\bullet was given in [PS08], 11.2.6 using Illusie's Koszul complex giving a (normalized) cohomological mixed Hodge complex $(C^\bullet, A^\bullet, \beta)$. The inclusion of W_0^Y gives an bifiltered injection of cohomological mixed Hodge complexes

$$\text{sp} : (W_0^Y C^\bullet, W_0^Y A^\bullet, W_0^Y \beta) \hookrightarrow (C^\bullet, A^\bullet, \beta)$$

whose cokernel we denote by $(\bar{C}^\bullet, \bar{A}^\bullet, \bar{\beta})$.

Theorem 4.5. (1) *We have an exact sequence of cohomological mixed Hodge complexes*

$$0 \rightarrow (W_0^Y C^\bullet, W_0^Y A^\bullet, W_0^Y \beta) \xrightarrow{\text{sp}} (C^\bullet, A^\bullet, \beta) \rightarrow (\bar{C}^\bullet, \bar{A}^\bullet, \bar{\beta}) \rightarrow 0.$$

(2) *The inclusion $W_0^Y A^\bullet \rightarrow A^\bullet$ is isomorphic to $\mathbb{C}_Y \rightarrow \psi_f \mathbb{C}_{\bar{X}}$ in $D^+(Y, \mathbb{Z})$ and thus \bar{A}^\bullet is isomorphic to $\phi_f \mathbb{C}_{\bar{X}}$. This gives a mixed Hodge structure on $\mathbb{H}^i(Y, \phi_f \mathbb{C}_{\bar{X}})$, and the sequence in (1) turns the long exact sequence*

$$\cdots \rightarrow H^i(Y, \mathbb{C}) \rightarrow \mathbb{H}^i(Y, \psi_f \mathbb{C}_{\bar{X}}) \rightarrow \mathbb{H}^i(Y, \phi_f \mathbb{C}_{\bar{X}}) \rightarrow H^{i+1}(Y, \mathbb{C}) \rightarrow \cdots$$

into an exact sequence of mixed Hodge structures.

(3) *We have $\text{Gr}_k^W \mathbb{H}^i(Y, \psi_f \mathbb{C}_{\bar{X}}) = \text{Gr}_k^W \mathbb{H}^i(Y, \phi_f \mathbb{C}_{\bar{X}})$ for $k \geq 2$.*

Proof. Part (1) follows from Lemma 4.4 and what we said before. The first part of (2) is given in the discussion after Theorem 11.28 of [PS08], the remainder of (2) is standard given (1). Since Y is compact, by [DelTH], III, 8.2.4, we have $h^{p,q} H^i(Y) = 0$ for $p + q > i$. This implies (3). \square

Remark 4.6. There is map a of cohomological mixed Hodge complexes $\text{Cone}_M(\text{sp}) \rightarrow (\bar{C}^\bullet, \bar{A}^\bullet, \bar{\beta})$ as is in the proof of Lemma 4.4, however it is not a filtered quasi-isomorphism even though it induces an isomorphism of mixed Hodge structures. The latter is cohomologically “more efficient” which is why we are using this rather than the cone.

Lemma 4.7. *We consider the spectral sequence of (\bar{A}^\bullet, W) , with*

$$(4.7) \quad E_1^{-k, m+k} : \mathbb{H}^m(X, \text{Gr}_k^W \bar{A}^\bullet) \Rightarrow \mathbb{H}^m(X, \bar{A}^\bullet).$$

We have

- (1) *The sequence (4.7) is degenerate at E_2 .*
- (2) *The Poincaré residue map along Y induces an isomorphism*

$$\text{Gr}_k^W \bar{A}^\bullet = \bigoplus_{q > -1, -k} \text{Gr}_{2q+k+1}^{W^Y} \Omega_X^\bullet(\log Y)[1] \xrightarrow{\sim} \bigoplus_{q > -1, -k} \Omega_{Y^{2q+k+1}}^\bullet[-2q - k].$$

(3) We thus have

$$\mathbb{H}^m(X, \mathrm{Gr}_k^W \bar{A}^\bullet) = \bigoplus_{q > -1, -k} H^{m-2q-k}(Y^{2q+k+1}, \mathbb{C}) \langle -q - k \rangle$$

where $\langle \cdot \rangle$ denotes the Tate twist.

(4) The map d_1 in (4.7) is given by $d_1 = \delta - \gamma$ where

$$\delta : H^l(Y^s, \mathbb{C}) \rightarrow H^l(Y^{s+1}, \mathbb{C})$$

is the restriction map given as

$$(\delta\alpha)|_{Y_{i_1} \cap \dots \cap Y_{i_s}} = \sum_j (-1)^{j+1} \alpha|_{Y_{i_1} \cap \dots \hat{Y}_{i_j} \cap \dots \cap Y_{i_s}},$$

$$\gamma : H^l(Y^s, \mathbb{C}) \rightarrow H^{l+2}(Y^{s-1}, \mathbb{C})$$

is the Gysin map, i.e., the Poincaré dual of δ .

(5) We have Poincaré duality for (4.7), i.e., if we set $n = \dim X$, $m' = 2n - m - 2$, $k' = 2 - k$, we have an isomorphism

$$E_1^{-k, m+k} = (E_1^{-k', m'+k'} \langle n \rangle)^*$$

which is compatible with the respective differentials d_1^* and d_1 . In particular, it also holds when we replace E_1 by E_∞ . We obtain

$$h^{p,q} \mathbb{H}^i(Y, \phi_f \mathbb{C}_{\bar{X}}) = h^{n-p, n-q} \mathbb{H}^{2n-2-i}(Y, \phi_f \mathbb{C}_{\bar{X}}).$$

(6) We have Poincaré duality for $E_1(A^\bullet, W)$ which yields

$$h^{p,q} \mathbb{H}^i(Y, \psi_f \mathbb{C}_{\bar{X}}) = h^{\dim Y - p, \dim Y - q} \mathbb{H}^{2 \dim Y - i}(Y, \psi_f \mathbb{C}_{\bar{X}}).$$

Proof. For (1) and (2), see, e.g., [PS08], Thm. 3.18 and §4.2, respectively. By (4.6), $F^i \mathrm{Gr}_k^W \bar{A}^\bullet$ becomes F^{i-q-k} on the right hand side of (2), thus the Tate twist in (3) becomes clear. We deduce (4) from [PS08], §11.3.2, p. 280. For (5), we apply Poincaré duality to each summand in (3). For Z a compact manifold, Poincaré duality means

$$H^i(Z, \mathbb{C}) = \mathrm{Hom}(H^{2 \dim Z - i}(Z, \mathbb{C}) \langle \dim Z \rangle, \mathbb{C}).$$

Using $\dim Y^i = n - i$, one remodels the resulting sum

$$E_1^{-k, m+k} = \left(\bigoplus_{q > -1, -k} H^{2n-2-m-2q-k}(Y^{2q+k+1}, \mathbb{C}) \langle n - q - 1 \rangle \right)^*$$

by replacing m, k, q by m', k' and $q' = q + k - 1$. Part (6) goes along the same lines as (5). \square

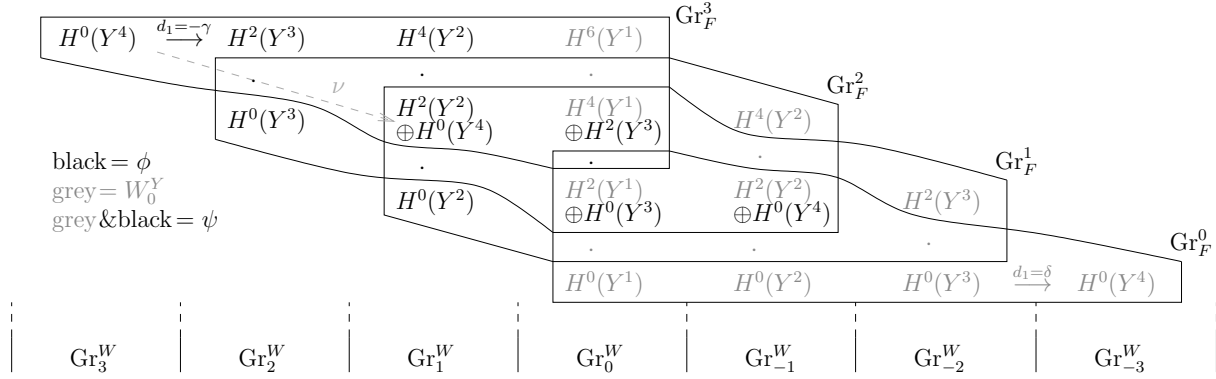


FIGURE 11. The E_1 term with respect to the weight filtration of the spectral sequence of the cohomology of the special fibre, nearby fibre and vanishing cycles for the case of a degeneration of a compact threefold, with odd cohomologies indicated by dots.

5. THE HODGE NUMBERS OF THE MIRROR

Given M , N , $M_{\mathbb{R}}$, $N_{\mathbb{R}}$, $\Delta \subseteq M_{\mathbb{R}}$, a star-like triangulation \mathcal{P} of Δ consisting only of standard simplices, we obtain data

$$w : X_{\Sigma} \rightarrow \mathbb{C}$$

$$\check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$$

with compactifications

$$\bar{w} : \tilde{\mathbb{P}}_{\Delta} \rightarrow \mathbb{P}^1$$

$$\bar{\check{w}} : \tilde{X}_{\check{\Sigma}} \rightarrow \mathbb{P}^1$$

given by Propositions 1.10 and 1.8 respectively. We choose a small disk $O \subseteq \mathbb{C} \subseteq \mathbb{P}^1$ with center $0 \in \mathbb{C}$ which does not contain any other critical values of \bar{w} or $\bar{\check{w}}$, and consider the restrictions

$$\bar{w} : \bar{w}^{-1}(O) \rightarrow O$$

$$\bar{\check{w}} : \bar{\check{w}}^{-1}(O) \rightarrow O$$

In the two cases, we have inclusions of open sets

$$j^D : X_{\Sigma} \cap \bar{w}^{-1}(O) \subseteq \bar{w}^{-1}(O),$$

$$\check{j}^D : X_{\check{\Sigma}} \cap \bar{\check{w}}^{-1}(O) \subseteq \bar{\check{w}}^{-1}(O).$$

We have already identified $\mathbb{H}^q(\check{w}^{-1}(0), \phi_{\bar{w}}(\mathbf{R}j_{*}^{\check{D}}\mathbb{C}_{X_{\Sigma}}))$ with $H^{q-1}(S, \mathbb{C})$ in Ex. 4.3. It is not hard to see that the usual Hodge structure from the Kähler manifold S and that from the vanishing cohomology construction, as given in the last section, coincide on $H^{q-1}(S, \mathbb{C})$. We can compute its Hodge numbers via the formulae in Prop. 3.2.

We are now going to compute the Hodge numbers of $\mathbb{H}^q(\bar{w}^{-1}(0), \phi_{\bar{w}}(\mathbf{R}j_*^D \mathbb{C}_{X_\Sigma}))$ in order to compare it to the former and to prove our main result. We apply the construction of the last section and use its notation, i.e., $\bar{X} = \bar{w}^{-1}(O)$, $X = X_\Sigma \cap \bar{X}$, $\bar{w} : \bar{X} \rightarrow O$,

$$Y = \bar{w}^{-1}(0) = \bigcup_{i=1}^{N_Y} Y_i \quad \text{and} \quad D = \bar{X} \setminus X = \bigcup_{i=1}^{N_D} D_i.$$

Indeed, by Lemma 1.21,(3) and simpliciality of $\bar{\Sigma}$, $Y \cup D$ is a normal crossing divisor.

In the definition (0.5) of the Hodge numbers for \check{S} , we throw out the information related to the weight filtration. Despite this, it is worth noting that the monodromy around 0 in the fibration defined by \bar{w} is related to the Kodaira dimension of S .

Proposition 5.1. *The logarithm of the monodromy ν operates on the hypercohomology of the vanishing cycles of the central fibre of w . Let m be the maximal integer such that $\nu^m \neq 0$ on cohomology. We have $m \leq \kappa(S)$ and the following are equivalent:*

- (1) $m = \dim S$
- (2) S is of general type (i.e. $\kappa(S) = \dim S$).

Proof. By Prop. 1.28 and Prop. 1.15, the dual intersection complex of $w^{-1}(0)$ is a $(\kappa(S)+1)$ -dimensional ball for $\kappa(S) < \dim S = d$ and a $(d+1)$ -dimensional sphere otherwise. This means that the largest k such that $Y^k \neq \emptyset$ is $\kappa(S) + 2$. Knowing the operation of ν on the expression given in Lemma 4.7,(3), we conclude $m \leq \kappa(S)$ which shows that (1) implies (2). For the converse, we need to show that the isomorphism $\nu^d : \mathbb{H}^\bullet(\mathrm{Gr}_{d+1}^W \bar{A}^\bullet) \xrightarrow{\sim} \mathbb{H}^\bullet(\mathrm{Gr}_{1-d}^W \bar{A}^\bullet)$, which is just $\mathrm{id} : H^0(Y^{d+2}) \xrightarrow{\sim} H^0(Y^{d+2})$, descends to a non-trivial map on cohomology with respect to d_1 . This map is

$$H_{d_1}(\nu^d) : \ker(-\gamma : H^0(Y^{d+2}) \rightarrow H^2(Y^{d+1})) \rightarrow \mathrm{coker}(\delta : H^0(Y^{d+1}) \rightarrow H^0(Y^{d+2})).$$

Since the intersection complex of $w^{-1}(0)$ is a $d+1$ -sphere, source and target are one-dimensional. Moreover, since γ is Poincaré dual to δ , it follows from the linear algebra property $\ker(f)^\perp = \mathrm{im}(f^*)$ of a linear map f and the identification of $H^0(Y^{d+2})$ with its Poincaré dual (turning it into an inner product space) that $H_{d_1}(\nu^d)$ is an isomorphism. \square

We now proceed to the main calculation. Motivated by Def. 3.1, we set

$$\begin{aligned} e^p(\check{S}, \mathcal{F}_{\check{S}}) &= (-1)^p \sum_{q,k} (-1)^q h^{p,q+k} \mathbb{H}^{p+q}(\check{S}, \mathcal{F}_{\check{S}}) \\ &= (-1)^p \sum_{q,k} (-1)^q h^{p+1,q+k} \mathbb{H}^{p+1+q}(Y, \bar{A}^\bullet). \end{aligned}$$

Recall that W_0 is the component of $w^{-1}(0)$ which is not contained in any toric stratum of X_Σ , i.e., the unique component of Y which meets D . Let $Y_{\mathrm{tor}}^i \subset Y^i$ be the subset of those components which are not contained in \tilde{W}_0 and $Y_{\mathrm{ntor}}^i = Y^i \setminus Y_{\mathrm{tor}}^i$. Note that $Y_{\mathrm{ntor}}^1 = \tilde{W}_0$. For $\tau \in \mathcal{P}$, we denote by $\mathcal{P}_*(\tau)$ the smallest cell of \mathcal{P}_* containing τ and by

$T_\tau \cong (\mathbb{C}^*)^{d+1-\dim \tau}$ the torus orbit in X_Σ corresponding to $\text{Cone}(\tau)$. Analogously, we define T_τ to be the torus orbit corresponding to $\tau \in \mathcal{P}_*$. Let $\mathcal{P}_{\Delta'}$, $\mathcal{P}_{\partial\Delta'}$ and $\mathcal{P}_{\partial\Delta}$ denote¹⁰ the induced subdivisions $\mathcal{P} \cap \Delta'$, $\mathcal{P} \cap \partial\Delta'$ and $\mathcal{P} \cap \partial\Delta$. Let $\mathcal{P}_{\partial\Delta'}^{[0]}$ denote the subset of vertices of $\mathcal{P}_{\partial\Delta'}$. For $\omega \in \mathcal{P}_{\Delta'}$, let X_ω denote the toric variety defined by the fan along ω . Note that $\dim Y^k = d+2-k$ and $\dim X_\omega = d+1-\dim \omega$, thus $X_\omega \subset Y^{\dim \omega+1}$.

Lemma 5.2. *We have*

- (1) $Y^k = Y_{\text{tor}}^k \sqcup Y_{\text{tor}}^{k-1} \cap \tilde{W}_0$, *i.e.*, $Y_{\text{ntor}}^k = Y_{\text{tor}}^{k-1} \cap \tilde{W}_0$,
- (2) $Y_{\text{tor}}^k = \coprod_{\substack{\omega \in \mathcal{P}_{\Delta'} \\ k=\dim \omega+1}} X_\omega$,
- (3) $X_\omega = \coprod_{\substack{\tau \in \mathcal{P} \\ \tau \supset \omega}} T_\tau$ for $\omega \in \mathcal{P}_{\Delta'}$,
- (4) $T_\tau \cap \tilde{W}_0 \cong (\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau)-\dim \tau} \times (T_{\mathcal{P}_*(\tau)} \cap W_0^*)$ for $\tau \in \mathcal{P}$.

Proof. (1) follows from Prop. 1.28 and (2)-(4) are standard in toric geometry where (4) uses the fact that w factors through $X_\Sigma \rightarrow X_{\Sigma^*}$. \square

5.1. The duality for the p -th Euler characteristic.

Lemma 5.3. (1) *For $\tau \subseteq \Delta'$, we have*

$$(-1)^{\dim \tau} = \sum_{\omega \in p_{\Delta\Delta'}^{-1}(\tau)} (-1)^{\dim \omega}.$$

(2) *For a polytope τ with a simplicial polyhedral decomposition \mathcal{P}_τ , we have*

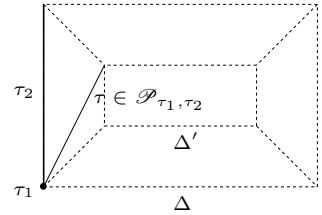
$$(-1)^{\dim \tau} = \sum_{\substack{\omega \in \mathcal{P}_\tau \\ \omega \not\subseteq \partial\tau}} (-1)^{\dim \omega}.$$

(3) *Let $\tau_1 \in \mathcal{P} \cap \partial\Delta$, $\tau_2 \subseteq \partial\Delta$ with $\mathcal{P}_*(\tau_1) \subseteq \tau_2$. We set*

$$\mathcal{P}_{\tau_1, \tau_2} = \{\tau \in \mathcal{P} \mid \tau \cap \partial\Delta = \tau_1, \tau \cap \Delta' \neq \emptyset, (p_{\Delta\Delta'}^1)^{-1}(\mathcal{P}_*(\tau)) = \tau_2\}$$

and have

$$\sum_{\tau \in \mathcal{P}_{\tau_1, \tau_2}} (-1)^{\dim \tau+1} = \begin{cases} (-1)^{\dim \tau_1} & \mathcal{P}_*(\tau_1) = \tau_2 \\ 0 & \mathcal{P}_*(\tau_1) \neq \tau_2. \end{cases}$$



Proof. (1) This is an Euler characteristic calculation. Following the notation of Lemma 1.19, let $\tilde{\tau} \in \tilde{\Sigma}_{\Delta'}$ be the cone dual to the face τ . Let $\tilde{\tau}'$ denote the inverse image of $\tilde{\tau}$ under the projection $N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\Delta'^{\perp}$. Then $\omega \in p_{\Delta\Delta'}^{-1}(\tau)$ if and only if the corresponding cone $\tilde{\omega} \in \tilde{\Sigma}_{\Delta}$ satisfies $\tilde{\omega} \subseteq \tilde{\tau}'$, $\tilde{\omega} \not\subseteq \partial\tilde{\tau}'$. Then

$$\sum_{\substack{\tilde{\omega} \in \tilde{\Sigma} \\ \tilde{\omega} \subseteq \tilde{\tau}', \tilde{\omega} \not\subseteq \partial\tilde{\tau}'}} (-1)^{\dim \tilde{\omega}} = \chi(\tilde{\tau}') - \chi(\partial\tilde{\tau}') = 1 - (1 + (-1)^{\dim \tilde{\tau}'-1}) = (-1)^{\dim \tilde{\tau}'}.$$

¹⁰We take $\partial\Delta'$ in the topology of Δ , e.g. $\partial\Delta' = \Delta'$ for $\dim \Delta' < \dim \Delta$.

Since $\dim \tilde{\omega} = \dim \Delta - \dim \omega$ and $\dim \tilde{\tau}' = (\dim \Delta - \dim \Delta') + (\dim \Delta' - \dim \tau)$, the desired result follows.

(2) As above, this is just a computation of the Euler characteristic of $\tau \setminus \partial\tau$.

(3) The proof will use Möbius inversion and be a variation of the proof of Lemma 3.5 in [KS10]. Recall that for any finite poset B , the incidence algebra consists of \mathbb{Z} -valued functions on $\{(a, b) | a, b \in B, a \leq b\}$ with the associative convolution product

$$(f * g)(a, b) = \sum_{a \leq x \leq b} f(a, x)g(x, b).$$

Its unit is δ which is non-zero only on $\{(a, a) | a \in B\}$ where it takes value one. If ζ denotes the function which is constant of value 1, then the Möbius function μ is its inverse, i.e.,

$$(5.1) \quad \delta = \zeta * \mu.$$

We set $\hat{B} = \{0\} \cup B$ and let $0 \leq a$ for all $a \in B$. For any function $h : B \rightarrow \mathbb{Z}$, we define $\hat{h} : \hat{B} \times \hat{B} \rightarrow \mathbb{Z}$ as $\hat{h}(a, b) = h(b)$ for $a = 0, b \in B$ and $\hat{h}(a, b) = 0$ otherwise. Multiplying (5.1) from the left by \hat{h} and restricting to $\{0\} \times B$ yields

$$(5.2) \quad h(b) = \sum_{x \leq b} \mu(x, b)g(x), \text{ where } g(x) = \sum_{a \leq x} h(a)$$

because $\hat{g} = \hat{h} * \zeta$. We apply this to our setup. First note that by Lemma 1.17, (3), we have for $\tau \in \mathcal{P}$ that

$$\tau \cap \Delta' \neq \emptyset \iff \tau \cap \text{Int}(\Delta) \neq \emptyset.$$

We pick $\tau'_1 \in \mathcal{P}_{\partial\Delta}$, $\hat{\tau}_2 \in \mathcal{P}_*$ with $\tau_1 \subseteq \tau'_1 \subseteq \hat{\tau}_2$ and $\hat{\tau}_2 \cap \text{Int}(\Delta) \neq \emptyset$. Let $\mathcal{P}|_{\hat{\tau}_2}$ denote the induced subdivision on $\hat{\tau}_2$. The link of $\tau'_1 \in \mathcal{P}|_{\hat{\tau}_2}$ is contractible and thus

$$\sum_{\tau \in \mathcal{P}|_{\hat{\tau}_2}, \tau'_1 \subsetneq \tau} (-1)^{\dim \tau - \dim \tau'_1 - 1} = 1,$$

and hence

$$\sum_{\tau \in \mathcal{P}|_{\hat{\tau}_2}, \tau \supseteq \tau'_1} (-1)^{\dim \tau + 1} = 0.$$

We think of this as the value of the function g at τ'_1 , where g is defined in (5.2) using the poset $\{\tau'_1 | \tau'_1 \in \mathcal{P}|_{\hat{\tau}_2}, \tau'_1 \subseteq \partial\Delta, \tau_1 \subseteq \tau'_1\}$ under reverse inclusion and the function

$$h(\tau'_1) = \sum_{\tau \in \mathcal{P}|_{\hat{\tau}_2}, \tau \cap \partial\Delta = \tau'_1} (-1)^{\dim \tau + 1}.$$

We then obtain as an expression for $h(\tau_1)$ the identity

$$(5.3) \quad \sum_{\tau \in \mathcal{P}|_{\hat{\tau}_2}, \tau \cap \partial\Delta = \tau_1} (-1)^{\dim \tau + 1} = 0.$$

Next consider the poset $B = \{\hat{\tau}_2 \mid \mathcal{P}_*(\tau_1) \subseteq \hat{\tau}_2, \hat{\tau}_2 \cap \text{Int}(\Delta) \neq \emptyset\}$ under inclusion which has a global minimal element $b_0 = p_{\Delta\Delta}^1(\mathcal{P}_*(\tau_1))$. We define $g : B \times B \rightarrow \mathbb{Z}$ by

$$g(b_0, \hat{\tau}_2) = \sum_{\substack{\tau \in \mathcal{P} \mid \hat{\tau}_2, \tau \cap \partial\Delta = \tau_1 \\ \tau \cap \text{Int}(\Delta) \neq \emptyset}} (-1)^{\dim \tau + 1},$$

which agrees with $(-1)^{\dim \tau_1}$ by (5.3), and we set $g(a, b) = 0$ for $a \neq b_0$. We are interested in

$$\sum_{\substack{\tau \in \mathcal{P} \mid \hat{\tau}_2, \tau \cap \partial\Delta = \tau_1 \\ \tau \cap \text{Int}(\hat{\tau}_2) \neq \emptyset}} (-1)^{\dim \tau + 1} = h(b_0, \hat{\tau}_2) = (g * \mu)(b_0, \hat{\tau}_2)$$

for $\hat{\tau}_2 = p_{\Delta\Delta}^1(\tau_2)$. However, on $\{b_0\} \times B$, we have $g = (-1)^{\dim \tau_1} \zeta$. By (5.1) we thus get

$$h(b_0, \hat{\tau}_2) = (-1)^{\dim \tau_1} \delta(b_0, \hat{\tau}_2)$$

which completes the proof. \square

Lemma 5.4. *For $\tau \in \mathcal{P}$, let T_τ denote the corresponding torus orbit in X_Σ . We have*

$$(-1)^p e^p(T_\tau) = (-1)^{d+1-\dim \tau} \binom{d+1-\dim \tau}{p}.$$

Moreover, for $\tau \not\subseteq \partial\Delta$, we have

$$(-1)^p e^p(T_\tau \cap \tilde{W}_0) = (-1)^{d-\dim \tau} \left(\binom{d+1-\dim \tau}{p+1} - \binom{\dim \mathcal{P}_*(\tau) - \dim \tau}{p+1} \right) + A_{\tau,p}.$$

Here, $A_{\tau,p} = 0$ if $\tau \not\subseteq \partial\Delta'$ and otherwise

$$A_{\tau,p} = \sum_{\hat{\tau} \in p_{\Delta\Delta'}^{-1}(\mathcal{P}_*(\tau))} (-1)^{\dim \mathcal{P}_*(\tau) - \dim \tau + d+1 - \dim \hat{\tau}} \left(\binom{\dim \hat{\tau} - \dim \tau}{p+1} - \binom{\dim \mathcal{P}_*(\tau) - \dim \tau}{p+1} \right).$$

Before we embark on the proof, note that the most simple form $T_\tau \cap \tilde{W}_0$ could have is a handlebody, i.e., the intersection of a general hyperplane with the open torus in the projective space. This occurs for $\dim \mathcal{P}_*(\tau) = \dim \tau$ and $A_{\tau,p} = 0$ in the above lemma. A slightly more complicated shape of $T_\tau \cap \tilde{W}_0$ is given for $\dim \mathcal{P}_*(\tau) \neq \dim \tau$ and $A_{\tau,p} = 0$ where it is a product of a lower-dimensional handlebody with an algebraic torus. Finally, $A_{\tau,p} \neq 0$ accounts for a $T_\tau \cap \tilde{W}_0$ which is a product of an algebraic torus with a decomposition of handlebodies rather than with a single handlebody.

Proof. By Lemma 3.4, we have $e^p((\mathbb{C}^*)^n) = (-1)^{p+n} \binom{n}{p}$. Since $\dim T_\tau = d+1-\dim \tau$, this proves the first statement. Note that $T_\tau \cap \tilde{W}_0 = \emptyset$ if $\dim \mathcal{P}_*(\tau) = d+1$ because $T_{\mathcal{P}_*(\tau)}$

is a point in that case. So we may assume that $\tau \not\subseteq \partial\Delta$ and $\dim \mathcal{P}_*(\tau) < d+1$. By Lemma 5.2, (4) and Thm. 3.3, (2),

$$(-1)^p e^p(T_\tau \cap \tilde{W}_0) = (-1)^p \sum_{k=0}^p e^k((\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau) - \dim \tau}) e^{p-k}(T_{\mathcal{P}_*(\tau)} \cap W_0^*).$$

By [DK86], 4.4, for $p \geq 0$, we have

$$(5.4) \quad e^p(T_{\mathcal{P}_*(\tau)} \cap W_0^*) = (-1)^{p+a_\tau-1} \binom{a_\tau}{p+1} + (-1)^{a_\tau-1} \varphi_{a_\tau-p}(\Delta_{\mathcal{P}_*(\tau)})$$

where $a_\tau = d+1 - \dim \mathcal{P}_*(\tau)$, $\Delta_{\mathcal{P}_*(\tau)} = \text{Newton}(T_{\mathcal{P}_*(\tau)} \cap W_0^*)$ and φ_i is defined as in the proof of Prop. 3.2. For $\mathcal{P}_*(\tau) \not\subseteq \Delta'$, by Lemma 1.19, (3), we have that $\Delta_{\mathcal{P}_*(\tau)}$ is a a_τ -dimensional standard simplex and thus $\varphi_i(\Delta_{\mathcal{P}_*(\tau)}) = 0$ for $i \leq a_\tau$. In this case, $T_\tau \cap \tilde{W}_0 \cong H^{a_\tau-1} \times (\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau) - \dim \tau}$, and Prop. 3.4 gives the result. The case $\tau \subseteq \partial\Delta'$ is similar, with the first term on the right hand side of (5.4) giving the same contribution as the previous case, and an additional possible contribution from $\varphi_{a_\tau-p}(\Delta_{\mathcal{P}_*(\tau)})$. We compute this term using Lemma 1.19 and the formula for φ_i given in the proof of Prop. 3.2 as follows:

$$\begin{aligned} & (-1)^p \sum_{k \geq 0} e^k((\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau) - \dim \tau}) (-1)^{a_\tau-1} \varphi_{a_\tau-p+k}(\Delta_{\mathcal{P}_*(\tau)}) \\ &= \sum_{k \geq 0} \binom{\dim \mathcal{P}_*(\tau) - \dim \tau}{k} \sum_{\hat{\tau} \in p_{\Delta\Delta'}^{-1}(\mathcal{P}_*(\tau))} (-1)^{m_{\tau,\hat{\tau}}} \binom{a_\tau - (d+1 - \dim \hat{\tau})}{a_\tau + 1 - (a_\tau - p + k)} \end{aligned}$$

where $(-1)^{m_{\tau,\hat{\tau}}} = (-1)^p (-1)^{k + \dim \mathcal{P}_*(\tau) - \dim \tau} (-1)^{a_\tau-1+a_\tau-p+k+(d+1-\dim \hat{\tau})+1}$ simplifies to $m_{\tau,\hat{\tau}} = \dim \mathcal{P}_*(\tau) - \dim \tau + (d+1 - \dim \hat{\tau})$. To obtain $A_{\tau,p}$, we need to subtract the $k = p+1$ term from the above which is

$$\sum_{\hat{\tau} \in p_{\Delta\Delta'}^{-1}(\mathcal{P}_*(\tau))} (-1)^{m_{\tau,\hat{\tau}}} \binom{\dim \mathcal{P}_*(\tau) - \dim \tau}{p+1}.$$

Using Prop. A.1, (2), yields $A_{\tau,p}$ as given in the assertion. \square

Recall from the introduction that $\mathcal{F}_{\check{S}} = \phi_{\bar{W},0} \mathbf{R}j_* \mathbb{C}_X[1]$ where the filtrations are shifted by

$$F^i \mathcal{F}_{\check{S}}^k = F^{i+1} \bar{A}^{k+1}, \quad W_i \mathcal{F}_{\check{S}}^k = W_{i+1} \bar{A}^{k+1}.$$

This implies

Lemma 5.5. $h^{p,q} \mathbb{H}^i(\check{S}, \mathcal{F}_{\check{S}}) = h^{p+1,q+1} \mathbb{H}^{i+1}(Y, \phi_{w,0} \mathbf{R}j_* \mathbb{C}_X).$

Theorem 5.6. *We have that*

(1) *Poincaré duality holds for $h^{p,q} \mathbb{H}^i(\check{S}, \mathcal{F}_{\check{S}})$, i.e.,*

$$h^{p,q} \mathbb{H}^i(\check{S}, \mathcal{F}_{\check{S}}) = h^{d-p,d-q} \mathbb{H}^{2d-i}(\check{S}, \mathcal{F}_{\check{S}}),$$

$$(2) \quad e^p(\check{S}, \mathcal{F}_{\check{S}}) = \sum_{i,j \geq 0} (-1)^{i+j} e^{p-i}(Y^{2+i+j}),$$

$$(3) \quad e^p(S) = (-1)^d e^{d-p}(\check{S}, \mathcal{F}_{\check{S}}).$$

Proof. (1) Using Lemma 4.7,(5), we get

$$\begin{aligned} h^{p,q} \mathbb{H}^i(\check{S}, \mathcal{F}_{\check{S}}) &= h^{p+1,q+1} \mathbb{H}^{i+1}(Y, \bar{A}^\bullet) \\ &= h^{d+2-(p+1), d+2-(q+1)} \mathbb{H}^{2d+2-(i+1)}(Y, \bar{A}^\bullet) = h^{d-p, d-q} \mathbb{H}^{2d-i}(\check{S}, \mathcal{F}_{\check{S}}). \end{aligned}$$

(2) The Euler characteristic can be computed as an alternating sum of dimensions of the terms in the E_1 term of (4.7). We use Lemma 4.7,(3) to get

$$\begin{aligned} &(-1)^p e^p(\check{S}, \mathcal{F}_{\check{S}}) \\ &= \sum_q (-1)^q \sum_k h^{p,q+k} \mathbb{H}^{p+q}(\check{S}, \mathcal{F}_{\check{S}}) \\ &= \sum_q (-1)^q \sum_k h^{p+1,q+k} \mathbb{H}^{p+1+q}(X, \text{Gr}_k^W \bar{A}^\bullet) \\ &= \sum_q (-1)^q \sum_k \sum_{q' > -1, -k} h^{p+1,q+k} H^{(p+1+q)-2q'-k}(Y^{2q'+k+1}, \mathbb{C}) \langle -q' - k \rangle \\ &= \sum_q (-1)^q \sum_k \sum_{q' > -1, -k} h^{p+1-q'-k, q-q'}(Y^{2q'+k+1}). \end{aligned}$$

Note that $\{(q', k) | q' > -1, -k\}$ and $\{(j, 1+i-j) | i, j \geq 0\}$ define the same subsets of \mathbb{Z}^2 , we may thus reorganize the sum via $k = 1+i-j, q' = j$ to get

$$\begin{aligned} (-1)^p e^p(\check{S}, \mathcal{F}_{\check{S}}) &= \sum_{q \geq 0} (-1)^q \sum_{i,j \geq 0} h^{p+1-j-(1+i-j), q-j}(Y^{2j+(1+i-j)+1}) \\ &= \sum_{i,j \geq 0} (-1)^{p-i+j} e^{p-i}(Y^{2+i+j}). \end{aligned}$$

(3) By (2) and Lemma 5.2,(1), we have

$$\begin{aligned} e^{d-p}(\check{S}, \mathcal{F}_{\check{S}}) &= \sum_{i,j \geq 0} (-1)^{i+j} e^{d-p-i}(Y^{2+i+j}) \\ &= \sum_{i,j \geq 0} (-1)^{i+j} e^{d-p-i}(Y_{\text{tor}}^{2+i+j}) \\ &\quad + \sum_{i,j \geq 0} (-1)^{i+j} e^{d-p-i}(Y_{\text{tor}}^{1+i+j} \cap \tilde{W}_0). \end{aligned}$$

Using Lemma 5.2,(2) and setting $2+i+j = \dim \omega + 1$ in the first and $1+i+j = \dim \omega + 1$ in the second sum allows us to continue the equality as

$$= \sum_{\substack{\omega \in \mathcal{P}_{\Delta'} \setminus \mathcal{P}_{\Delta'}^{[0]} \\ 0 \leq i \leq \dim \omega - 1}} (-1)^{\dim \omega - 1} e^{d-p-i}(X_\omega) + \sum_{\substack{\omega \in \mathcal{P}_{\Delta'} \\ 0 \leq i \leq \dim \omega}} (-1)^{\dim \omega} e^{d-p-i}(X_\omega \cap \tilde{W}_0)$$

and by Lemma 5.2,(3), as

$$= \sum_{\substack{\tau \in \mathcal{P} \\ \omega \subseteq \tau \cap \Delta' \\ 0 \leq i \leq \dim \omega - 1}} (-1)^{\dim \omega - 1} e^{d-p-i}(T_\tau) + \sum_{\substack{\tau \in \mathcal{P} \\ \omega \subseteq \tau \cap \Delta' \\ -1 \leq i \leq \dim \omega - 1}} (-1)^{\dim \omega} e^{d-p-i-1}(T_\tau \cap \tilde{W}_0).$$

Note that, using Prop. A.1,(1), for any simplex τ and $i \geq -1$, we have

$$\sum_{\substack{\omega \subseteq \tau \\ \dim \omega \geq i+1}} (-1)^{\dim \omega} = \sum_{j=i+1}^{\dim \tau} (-1)^j \binom{\dim \tau + 1}{j+1} = (-1)^{i+1} \binom{\dim \tau}{i+1}$$

which we insert above to have

$$(5.5) \quad \begin{aligned} e^{d-p}(\check{S}, \mathcal{F}_{\check{S}}) &= \sum_{\substack{\tau \in \mathcal{P} \\ 0 \leq i \leq \dim \tau \cap \Delta' - 1}} (-1)^i \binom{\dim \tau \cap \Delta'}{i+1} e^{d-p-i}(T_\tau) \\ &+ \sum_{\substack{\tau \in \mathcal{P} \\ -1 \leq i \leq \dim \tau \cap \Delta' - 1}} (-1)^{i+1} \binom{\dim \tau \cap \Delta'}{i+1} e^{d-p-i-1}(T_\tau \cap \tilde{W}_0). \end{aligned}$$

We apply Lemma 5.4 and Prop. A.1,(2), and obtain

$$(-1)^p e^{d-p}(\check{S}, \mathcal{F}_{\check{S}}) = C_1 + C_2 + C_3 + C_4$$

where

$$\begin{aligned} C_1 &= \sum_{\substack{\tau \in \mathcal{P} \\ \tau \cap \Delta' \neq \emptyset}} (-1)^{\dim \tau + 1} \binom{d+1 - \dim \tau + \dim \tau \cap \Delta'}{d-p+1} \\ C_2 &= \sum_{\substack{\tau \in \mathcal{P} \\ \tau \cap \Delta' \neq \emptyset}} (-1)^{\dim \tau} \binom{d+1 - \dim \tau + \dim \tau \cap \Delta'}{d-p+1} \\ &\quad - \sum_{\substack{\tau \in \mathcal{P} \\ \tau \cap \Delta' \neq \emptyset \\ \dim \mathcal{P}_*(\tau) < d+1}} (-1)^{\dim \tau} \binom{\dim \mathcal{P}_*(\tau) - \dim \tau + \dim \tau \cap \Delta'}{d-p+1} \\ C_3 &= \sum_{\substack{\tau \in \mathcal{P} \\ \mathcal{P}_*(\tau) \subseteq \partial \Delta' \\ \hat{\tau} \in p_{\Delta \Delta'}^{-1}(\mathcal{P}_*(\tau))}} (-1)^{\dim \mathcal{P}_*(\tau) - \dim \tau + 1 - \dim \hat{\tau}} \left(\binom{\dim \hat{\tau}}{d-p+1} - \binom{\dim \mathcal{P}_*(\tau)}{d-p+1} \right) \\ C_4 &= \sum_{\substack{\tau \in \mathcal{P} \\ \tau \cap \Delta' \neq \emptyset}} (-1)^{\dim \tau} \binom{d+1 - \dim \tau}{d-p+1} \\ &\quad + \sum_{\substack{\tau \in \mathcal{P} \\ \tau \cap \Delta' \neq \emptyset \\ \dim \mathcal{P}_*(\tau) = d+1}} (-1)^{\dim \tau + 1} \binom{d+1 - \dim \tau + \dim \tau \cap \Delta'}{d-p+1}. \end{aligned}$$

Here C_1 is the first term of the right-hand-side of (5.5), along with an additional contribution for $i = -1$; this latter contribution is cancelled by the first term of C_4 . The expression for C_2 comes from the second term of (5.5), using Lemma 5.4, without taking into account the term $A_{\tau,p}$ in that lemma. However, the first sum of C_2 includes a contribution from cells τ with $\dim \mathcal{P}_*(\tau) = d+1$, for which $e^{d-p-i-1}(T_\tau \cap \tilde{W}_0) = 0$. The second term in C_4 cancels this contribution. Finally, the $A_{\tau,p}$ term is accounted for in C_3 .

The first sum of C_2 cancels with C_1 , the deeper reason for this being the Lefschetz hyperplane theorem. Using Lemma 5.3,(2), the second sum of C_2 can be written as

$$\begin{aligned} & \sum_{\tau \in \mathcal{P}_{\partial \Delta'}} (-1)^{\dim \tau + 1} \binom{\dim \mathcal{P}_*(\tau)}{d-p+1} + C'_2 \\ &= \sum_{\tau \subset \partial \Delta'} (-1)^{\dim \tau + 1} \binom{\dim \tau}{d-p+1} + C'_2 \end{aligned}$$

where

$$C'_2 = \sum_{\substack{\tau \in \mathcal{P} \setminus \mathcal{P}_{\Delta'} \\ \tau \cap \Delta' \neq \emptyset \\ \dim \mathcal{P}_*(\tau) < d+1}} (-1)^{\dim \tau + 1} \binom{\dim \mathcal{P}_*(\tau) - \dim \tau + \dim \tau \cap \Delta'}{d-p+1}$$

We apply Lemma 5.3,(2) and (1) successively to C_3 to get

$$\begin{aligned} C_3 &= \sum_{\substack{\omega \subseteq \partial \Delta' \\ \tau \in p_{\Delta \Delta'}^{-1}(\omega)}} (-1)^{\dim \tau + 1} \left(\binom{\dim \tau}{d-p+1} - \binom{\dim \omega}{d-p+1} \right) \\ &= \sum_{\tau \subseteq \Delta} (-1)^{\dim \tau + 1} \binom{\dim \tau}{d-p+1} + \sum_{\tau \subset \partial \Delta'} (-1)^{\dim \tau} \binom{\dim \tau}{d-p+1} \\ &\quad + \delta_{\dim \Delta}^{\dim \Delta'} (-1)^{\dim \Delta} \binom{\dim \Delta}{d-p+1} \end{aligned}$$

where δ denotes the Kronecker symbol. This last term arises because if $\dim \Delta' = \dim \Delta$, then $\partial \Delta' \neq \Delta'$, and hence $\Delta \notin p_{\Delta \Delta'}^{-1}(\omega)$ for any $\omega \subseteq \partial \Delta'$. Using Lemma 5.3,(2), we rewrite the part of the second sum of C_4 involving those τ with $\tau \subseteq \Delta'$ as

$$\sum_{\substack{\tau \in \mathcal{P}_{\Delta'} \\ \dim \mathcal{P}_*(\tau) = d+1}} (-1)^{\dim \tau + 1} \binom{d+1 - \dim \tau + \dim \tau \cap \Delta'}{d-p+1} = \delta_{\dim \Delta}^{\dim \Delta'} (-1)^{\dim \Delta' + 1} \binom{\dim \Delta'}{d-p+1}.$$

Putting all transformations together in the previous order, after term pair cancellations in (C_1, C_2) , (C_2, C_3) and (C_3, C_4) , we obtain

$$\begin{aligned}
(-1)^p e^{d-p}(\check{S}, \mathcal{F}_{\check{S}}) = & \sum_{\substack{\tau \in \mathcal{P} \setminus \mathcal{P}_{\Delta'} \\ \tau \cap \Delta' \neq \emptyset \\ \dim \mathcal{P}_*(\tau) < d+1}} (-1)^{\dim \tau + 1} \binom{\dim \mathcal{P}_*(\tau) - \dim \tau + \dim \tau \cap \Delta'}{d - p + 1} \\
& + \sum_{\tau \subseteq \Delta} (-1)^{\dim \tau + 1} \binom{\dim \tau}{d - p + 1} \\
& + \sum_{\substack{\tau \in \mathcal{P} \\ \tau \cap \Delta' \neq \emptyset}} (-1)^{\dim \tau} \binom{d + 1 - \dim \tau}{d - p + 1} \\
& + \sum_{\substack{\tau \in \mathcal{P} \setminus \mathcal{P}_{\Delta'} \\ \tau \cap \Delta' \neq \emptyset \\ \dim \mathcal{P}_*(\tau) = d+1}} (-1)^{\dim \tau + 1} \binom{d + 1 - \dim \tau + \dim \tau \cap \Delta'}{d - p + 1}.
\end{aligned}$$

Recall that $\Delta(\tau)$ denotes the smallest face of Δ containing $\tau \in \mathcal{P}$. Note that since Δ' contains all lattice points in the interior of Δ , $\dim \Delta(\tau) = d + 1$ is equivalent to $\tau \cap \Delta' \neq \emptyset$, so the third sum becomes

$$\sum_{\substack{\tau \in \mathcal{P} \\ \tau \not\subseteq \partial \Delta}} (-1)^{\dim \tau} \binom{\dim \Delta(\tau) - \dim \tau}{d - p + 1}.$$

For $\tau \in \mathcal{P} \setminus \mathcal{P}_{\Delta'}$ with $\tau \cap \Delta' \neq \emptyset$, we have $\dim \tau = \dim \tau \cap \partial \Delta + \dim \tau \cap \Delta' + 1$. We can unite the first and fourth sum and write this as

$$\begin{aligned}
& \sum_{\substack{\tau \in \mathcal{P} \setminus \mathcal{P}_{\Delta'} \\ \tau \cap \Delta' \neq \emptyset}} (-1)^{\dim \tau + 1} \binom{\dim \mathcal{P}_*(\tau) - \dim \tau \cap \partial \Delta - 1}{d - p + 1} \\
(5.6) \quad & = \sum_{\substack{\tau' \in \mathcal{P}_{\partial \Delta} \\ \hat{\tau} \in \mathcal{P}_*, \hat{\tau} \supseteq \mathcal{P}_*(\tau') \\ \hat{\tau} \cap \Delta' \neq \emptyset}} \binom{\dim \hat{\tau} - \dim \tau' - 1}{d - p + 1} \sum_{\substack{\tau \in \mathcal{P} \\ \mathcal{P}_*(\tau) = \hat{\tau} \\ \tau \cap \partial \Delta = \tau'}} (-1)^{\dim \tau + 1}.
\end{aligned}$$

In order to apply Lemma 5.3,(3), we identify

$$\mathcal{P}_{\tau', (p_{\Delta \Delta'}^1)^{-1}(\hat{\tau})} = \{\tau \in \mathcal{P} \mid \mathcal{P}_*(\tau) = \hat{\tau}, \tau \cap \partial \Delta = \tau'\}$$

and obtain

$$\sum_{\substack{\tau \in \mathcal{P} \\ \mathcal{P}_*(\tau) = \hat{\tau} \\ \tau \cap \partial \Delta = \tau'}} (-1)^{\dim \tau + 1} = \begin{cases} (-1)^{\dim \tau'} & \mathcal{P}_*(\tau') = (p_{\Delta \Delta'}^1)^{-1}(\hat{\tau}) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the non-trivial case coincides with $\hat{\tau} = p_{\Delta \Delta'}^1(\mathcal{P}_*(\tau'))$ such that the sum on the right-hand-side of (5.6) can be reduced to a sum over $\tau' \in \mathcal{P}_{\partial \Delta}$ by using $\dim p_{\Delta \Delta'}^1(\mathcal{P}_*(\tau')) =$

$\dim \mathcal{P}_*(\tau') + 1$. Identifying $\mathcal{P}_*(\tau') = \Delta(\tau')$ for $\tau' \in \mathcal{P}_{\partial\Delta}$ and comparing the results with Prop. 3.2,(2), we get

$$(-1)^{d-p} e^{d-p}(S) = (-1)^p e^{d-p}(\check{S}, \mathcal{F}_{\check{S}})$$

and by Poincaré duality for S , we have $e^{d-p}(S) = e^p(S)$ which finishes the proof. \square

5.2. A vanishing result. By the Lefschetz hyperplane theorem, $h^{p,q}(S) = 0$ unless $p = q$ or $p + q = d$. In this section, we prove that the corresponding mirror dual Hodge numbers also vanish. We recall the notation from (1.6). We will often drop the coefficient ring \mathbb{C} from a cohomology group, writing $H^k(T)$ instead of $H^k(T, \mathbb{C})$ for a variety T .

Theorem 5.7. *Let Z be a smooth hypersurface in $(\mathbb{C}^*)^k$ given by a Laurent polynomial whose Newton polytope is k -dimensional. Then $h^{p,q}H^i(Z) = 0$ unless either $i < \dim Z$ and $i = 2p = 2q$ or $i = \dim Z$ and $p + q \geq i$.*

Proof. For a smooth affine variety, $H^i(Z) = 0$ for $i > \dim Z$ anyway. For a smooth variety Z , $h^{p,q}H^i(Z) = 0$ for $p + q < i$ (see e.g., [PS08], Thm. 5.39). The remaining statements follow from the Lefschetz-type theorem of [DK86], Prop. 3.9. \square

Let \tilde{D} denote the complement of the dense torus in $\tilde{\mathbb{P}}_{\tilde{\Delta}}$. By Lemma 1.21,(3) and the simpliciality of $\tilde{\Sigma}$, $\tilde{W}_0 \cap \tilde{D}$ is a normal crossing divisor in \tilde{W}_0 . Let $\delta^{\tilde{W}_0 \cap \tilde{D}} : H^k(\tilde{W}_0 \cap \tilde{D}^i) \rightarrow H^k(\tilde{W}_0 \cap \tilde{D}^{i+1})$ denote the differential and augmentation of the cohomological complex associated to the semi-simplicial scheme $\tilde{W}_0 \cap \tilde{D}$ and let $\gamma^{\tilde{W}_0 \cap \tilde{D}}$ be its Poincaré dual, the Gysin map.

Lemma 5.8. (1) *There is a sequence*

$$\dots \rightarrow H^{p-i, q-i}(\tilde{W}_0 \cap \tilde{D}^i) \xrightarrow{-\gamma^{\tilde{W}_0 \cap \tilde{D}}} \dots \xrightarrow{-\gamma^{\tilde{W}_0 \cap \tilde{D}}} H^{p-1, q-1}(\tilde{W}_0 \cap \tilde{D}^1) \xrightarrow{-\gamma^{\tilde{W}_0 \cap \tilde{D}}} H^{p, q}(\tilde{W}_0) \rightarrow 0.$$

For $p \neq q$ and $\dim \Delta' > 0$, the sequence is exact at every term except possibly at $H^{p-i, q-i}(\tilde{W}_0 \cap \tilde{D}^i)$ where $p + q - 2i = \dim \tilde{W}_0 \cap \tilde{D}^i = d + 1 - i$.

(2) *For $p \neq q$, there is a sequence*

$$\dots \rightarrow H^{p-i, q-i}(\tilde{W}_0 \cap Y_{\text{tor}}^i) \xrightarrow{-\gamma^{Y_{\text{tor}}}} \dots \xrightarrow{-\gamma^{Y_{\text{tor}}}} H^{p-1, q-1}(\tilde{W}_0 \cap Y_{\text{tor}}^1) \xrightarrow{-\gamma^{Y_{\text{tor}}}} H^{p, q}(\tilde{W}_0)$$

where each map is an alternating sum of Gysin maps given by projecting $-\gamma$ in Lemma 4.7,(4), to Y_{tor}^\bullet . When replacing the last term by the image of the last map, the resulting sequence is a direct summand of the sequence in (1) and thus it is exact at every term except possibly at $H^{p-i, q-i}(\tilde{W}_0 \cap Y_{\text{tor}}^i)$ where $p + q - 2i = \dim \tilde{W}_0 \cap Y_{\text{tor}}^i = d + 1 - i$. Moreover, if $\dim \Delta' = 0$ then it is exact everywhere for every p, q .

Proof. The sequence in (1) can be derived from the weight spectral sequence of the cohomological mixed Hodge complex with complex part $\Omega_{\tilde{W}_0}^\bullet(\log(\tilde{W}_0 \cap \tilde{D}))$ computing the mixed Hodge structure on $H^{a+b}(\tilde{W}_0 \setminus (\tilde{W}_0 \cap \tilde{D}))$. It is

$$(5.7) \quad E_1^{a,b} = \mathbb{H}^{a+b}(\tilde{W}_0, \mathrm{Gr}_{-a}^{W_{\tilde{D}}} \Omega_{\tilde{W}_0}^\bullet(\log(\tilde{W}_0 \cap \tilde{D}))) \Rightarrow H^{a+b}(\tilde{W}_0 \setminus (\tilde{W}_0 \cap \tilde{D})).$$

Using the residue map, in terms of the fan $\bar{\Sigma}$, this becomes

$$E_1^{a,b} = \bigoplus_{\substack{\tau \in \bar{\Sigma} \\ \dim \tau = -a}} H^{2a+b}(\tilde{V}(\tau) \cap \tilde{W}_0),$$

where $V(\tau)$ denotes the closure of the orbit corresponding to τ and $\tilde{V}(\tau)$ denotes its inverse image under the blowup $\tilde{\mathbb{P}}_{\tilde{\Delta}} \rightarrow X_{\tilde{\Sigma}}$. The differential $d_1 = -\gamma^{\tilde{W}_0 \cap \tilde{D}}$ is given explicitly in [PS08], Prop. 4.10 as the (twisted) Gysin map. Setting $a = -i, b = p+q$ gives the sequence in the assertion. By Lemma 1.20, we have

$$\dim \check{\Delta}_0 \neq d+2 \iff \dim \check{\Delta}_0 = d+1 \iff \dim \Delta' = 0,$$

so we assume $\dim \check{\Delta}_0 = d+2$. We have

$$E_\infty^{a,b} = E_2^{a,b} = \mathrm{Gr}_{-a}^W H^{a+b}(\tilde{W}_0 \setminus (\tilde{W}_0 \cap \tilde{D})).$$

The exactness follows if we show that

$$(5.8) \quad h^{p',q'} \mathrm{Gr}_{-a}^W H^{a+b}(\tilde{W}_0 \setminus (\tilde{W}_0 \cap \tilde{D})) = 0 \text{ for } p' \neq q'$$

unless $a+b = d+1$. This follows directly from $\tilde{W}_0 \setminus (\tilde{W}_0 \cap \tilde{D}) = \bar{W}_0 \setminus (\bar{W}_0 \cap \bar{D})$ and Thm. 5.7, where \bar{D} is the toric boundary in $X_{\bar{\Sigma}}$.

To prove (2), we set

$$A = \{\tau \in \bar{\Sigma} \mid \tau = \mathrm{Cone}(\tau_1), \tau_1 \in \mathcal{P}, \tau_1 \not\subseteq \Delta', \tau_1 \not\subseteq \partial\Delta\}.$$

Note that $\tilde{V}(\tau) = V(\tau)$ for $\tau \in A$. Prop. 1.21,(2), implies that for $\tau \in A$, $\tilde{W}_0 \cap V(\tau)$ is the pullback of a projective space under a toric blowup. Hence $H^{p,q}(\tilde{W}_0 \cap V(\tau)) = 0$ for $p \neq q$ and $\tau \in A$. Moreover, $\mathrm{supp} \bar{\Sigma} \setminus \mathrm{supp}(\{\mathrm{Int}(\tau) \mid \tau \in A\} \cup \{0\})$ has two connected components. We focus on the component of cones contained in $\mathrm{Cone}(\Delta')$. As a summand of the sequence in (1), we get the desired sequence

$$\cdots \rightarrow \bigoplus_{\substack{\tau \in \bar{\Sigma} \cap \mathrm{Cone}(\Delta') \\ \dim \tau = i}} H^{p-i, q-i}(V(\tau) \cap \tilde{W}_0) \rightarrow \cdots \rightarrow \bigoplus_{\substack{\tau \in \bar{\Sigma} \cap \mathrm{Cone}(\Delta') \\ \dim \tau = 1}} H^{p-1, q-1}(V(\tau) \cap \tilde{W}_0) \rightarrow H^{p,q}(\tilde{W}_0)$$

by identifying $Y_{\mathrm{tor}}^i = \coprod_{\substack{\tau \in \bar{\Sigma} \cap \mathrm{Cone}(\Delta') \\ \dim \tau = i}} V(\tau)$.

We now treat the case $\dim \Delta' = 0$ separately by a different proof. The projection $\pi_{\mathrm{Cone}(\Delta')}$ in Prop. 1.10,(2) induces a projection

$$\bar{W}_0 \rightarrow \bar{W}_0 \cap D_{\mathrm{Cone}(\Delta')} = Y_{\mathrm{ntor}}^2$$

for which the inclusion $Y_{\text{ntor}}^2 \rightarrow \bar{W}_0$ provides a section. Hence, the pullback by the inclusion is surjective on cohomology and thus its dual, the Gysin map $H^{i-2}(Y_{\text{ntor}}^2) \rightarrow H^i(\bar{W}_0)$, is an injection. Since $Y_{\text{ntor}}^1 = \tilde{W}_0$ is a blowup of \bar{W}_0 , we may compose the above injection with the injection $H^i(\bar{W}_0) \rightarrow H^i(Y_{\text{ntor}}^1)$. This composition is indeed $-\gamma^{Y_{\text{ntor}}}$. \square

Proposition 5.9. *We have*

- (1) $h^{p,q+k} \mathbb{H}^{p+q}(\check{S}, \mathcal{F}_{\check{S}}) = h^{p+1,q+k+1} \mathbb{H}^{p+q+1}(Y, \psi_{\bar{w},0} \mathbb{C}_{\bar{X}})$ for $k \geq 1$,
- (2) $h^{p,q+k} \mathbb{H}^{p+q}(Y, \psi_{\bar{w},0} \mathbb{C}_{\bar{X}}) = 0$ unless $p+q = d+1$ or $k = 0$,
- (3) $h^{p,q+k} \mathbb{H}^{p+q}(\check{S}, \mathcal{F}_{\check{S}}) = 0$ unless $p+q = d$ or $p-q = k = 0$.

Proof. (1) follows from Lemma 5.5 and Thm. 4.5,(3). By Poincaré duality, Lemma 4.7,(6), it suffices to prove the vanishing in (2) for $p+q > d+1$ and $k \neq 0$. Choose $t_0 \in \mathbb{C}$ with $|t_0|$ sufficiently small so that 0 is the only critical value of \bar{w} for in the closed disk with radius $|t_0|$. Note that $\bar{W}_{t_0} := \bar{w}^{-1}(t_0)$ is a $\bar{\Sigma}$ -regular hypersurface as argued in the proof of Prop. 1.10. As in the proof of Lemma 5.8,(1), we have an exact sequence

$$\bigoplus_{\substack{\tau \in \Sigma \\ \dim \tau = 1}} H^{b-2}(\tilde{V}(\tau) \cap \tilde{W}_{t_0}) \rightarrow H^b(\tilde{W}_{t_0}) \rightarrow \text{Gr}_0^W H^b(\bar{W}_{t_0} \setminus (\bar{W}_{t_0} \cap \bar{D})) \rightarrow 0.$$

Let T denote the monodromy operator for \bar{w} around 0. Then T and $N = \log T$ operate on this sequence. Note that it suffices to show that N is trivial on $H^b(\tilde{W}_{t_0})$ for $b > d+1$ because then the (monodromy) weight filtration on $\mathbb{H}^b(Y, \psi_{\bar{w},0} \mathbb{C}_{\bar{X}})$ is also trivial, i.e.,

$$\mathbb{H}^b(Y, \psi_{\bar{w},0} \mathbb{C}_{\bar{X}}) = \text{Gr}_0^W \mathbb{H}^b(Y, \psi_{\bar{w},0} \mathbb{C}_{\bar{X}}).$$

By Thm. 5.7, $\text{Gr}_0^W H^b(\bar{W}_{t_0} \setminus (\bar{W}_{t_0} \cap \bar{D})) = 0$ for $b > d+1$, so we only need to show that N is trivial on $H^{b-2}(\tilde{V}(\tau) \cap \tilde{W}_{t_0})$. It suffices to show the triviality on $H^{b-2}(V(\tau) \cap \bar{W}_{t_0})$. We show that $\bar{w}|_{V(\tau)}$ is constant if $\tau \notin \Sigma$. Recall that the pencil defined by \bar{w} as a family of sections of $\phi^* \mathcal{O}_{\mathbb{P}_{\bar{\Delta}}}(1)$ (where $\phi : X_{\bar{\Sigma}} \rightarrow \mathbb{P}_{\bar{\Delta}}$ is the resolution) is

$$\bar{w}(t) = t \cdot z^0 + c_{\rho} z^{\rho} + \sum_{\omega \subset \Delta} c_{\omega} z^{(n_{\omega}, \varphi_{\Delta}(n_{\omega}))}$$

and z^0 vanishes on $D_{\infty} = X_{\bar{\Sigma}} \setminus X_{\Sigma}$. We have $V(\tau) \subseteq D_{\infty}$ if $\tau \notin \Sigma$, so indeed \bar{w} is constant on such $V(\tau)$. Now let us assume that $\tau \in \Sigma \setminus \{0\}$. The Newton polytope of $\bar{W}_t^{\tau} := \bar{w}^{-1}(t) \cap V(\tau)$ is a proper face of $\check{\Delta}$ supported by the hyperplane τ^{\perp} . It contains 0 and thus by the smoothness assumption of $\mathbb{P}_{\bar{\Delta}}$, this face generates the standard cone $\tau^{\perp} \cap \check{\sigma} \subseteq \partial \check{\sigma}$. For each t , there is a diagram

$$\begin{array}{ccccc} X_{\bar{\Sigma}} & \hookleftarrow & V(\tau) & \hookleftarrow & \bar{W}_t^{\tau} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}_{\bar{\Delta}} & \hookleftarrow & \mathbb{P}_{\bar{\Delta} \cap \tau^{\perp}} & \hookleftarrow & W_t^{\tau} \end{array}$$

where the first vertical map is birational, the second and third vertical maps have toric fibres, the horizontal maps are closed embeddings, the squares are pullback diagrams, $\mathbb{P}_{\tilde{\Delta} \cap \tau^\perp} \cong \mathbb{P}^{\dim \tilde{\Delta} \cap \tau^\perp}$ and W_t^τ is a hyperplane section in the latter. The fibres of the vertical maps over closed points of W_t^τ are toric varieties. In particular, $W_{t_0}^\tau$ is a $\tilde{\Delta} \cap \tau^\perp$ -regular hypersurface and thus has a disjoint decomposition in handlebodies of different dimensions induced from the intersection with the toric strata in $\mathbb{P}_{\tilde{\Delta} \cap \tau^\perp}$. Since handlebodies as well as toric varieties have Hodge structures concentrated in degrees (p, q) with $p = q$ (see Lemma 3.4), this also holds for $\bar{W}_{t_0}^\tau$ which inherits a decomposition in products of handlebodies and toric varieties. The monodromy theorem, e.g., [PS08], Cor. 11.42, implies that N operates trivially on $H^\bullet(\bar{W}_{t_0}^\tau)$.

We now show (3). Note that (1) and (2) and the Poincaré duality of Lemma 4.7,(5), imply the vanishing for $k \neq 0$. It suffices to show it for the case where $k = 0, p + q > d$ and $p \neq q$. We use Lemma 5.5 and work with \bar{A}^\bullet , i.e., we want to show

$$h^{p+1, q+1} \mathbb{H}^{p+q+1}(Y, \bar{A}^\bullet) = 0$$

for $p + q > d$. Recall that Lemma 4.7 provides us with a sequence

$$\dots \rightarrow {}_W E_1^{-k, m+k} \xrightarrow{d_1} {}_W E_1^{-(k-1), (m+1)+(k-1)} \rightarrow \dots$$

which becomes

$$\dots \rightarrow \bigoplus_{\tilde{q} > -1, -k} H^{m-2\tilde{q}-k}(Y^{2\tilde{q}+k+1}) \xrightarrow{d_1} \bigoplus_{\tilde{q} > -1, -(k-1)} H^{m-2\tilde{q}-k+2}(Y^{2\tilde{q}+k}) \rightarrow \dots$$

We have $\dim Y^i = d+2-i$, so $H^j(Y^i) = 0$ for $2i+j > 2d+4$ and in particular $H^{m-1}(Y^i) = 0$ for $i > d+2-(m-1)/2$. We fix m . Because d_1 splits up as $d_1 = \delta - \gamma$, the above sequence is the total complex of the double complex

$$\begin{array}{ccccccc} H^{m-1}(Y^2) & \xrightarrow{\delta} & \dots & \longrightarrow & H^{m-1}(Y^{d+1-(m-1-i)/2}) & \longrightarrow & H^{m-1}(Y^{d+2-(m-1-i)/2}) \\ \uparrow -\gamma & \searrow k=1 & \uparrow & & \searrow k=2-d+(m-1-i)/2 & & \uparrow \\ H^{m-3}(Y^3) & \longrightarrow & H^{m-3}(Y^4) & \longrightarrow & \dots & \longrightarrow & H^{m-3}(Y^{d+3-(m-1-i)/2}) \\ \uparrow & & \uparrow & & & & \uparrow \\ \vdots & & \vdots & & & & \vdots \\ \uparrow -\gamma & \searrow k=(m-1-i)/2 & \uparrow & & \searrow k=1-d+(m-1-i) & & \uparrow \\ H^i(Y^{2+(m-1-i)/2}) & \xrightarrow{\delta} & H^i(Y^{3+(m-1-i)/2}) & \longrightarrow & \dots & \longrightarrow & H^i(Y^{d+2}) \end{array}$$

concentrated in a rectangle and with $i = 1$ if m is even and $i = 0$ otherwise. Here the main diagonal (marked as $k = 1$) gives $E^{-1, m+1}$, with other diagonals giving $E^{-k, (m-k+1)+k}$ for various k . We set $m = p + q + 1 > d + 1$. Note that $\text{Gr}_1^W \mathbb{H}^{p+q+1}(Y, \bar{A}^\bullet)$ is the cohomology group by the total differential at the main diagonal. Since $m > d + 1$, the rectangle extends more in the $-\gamma$ -direction than it does in the δ -direction. We restrict this double complex

to the off-diagonal Hodge classes, i.e., we write $\bigoplus_{p' \neq q'} H^{p', q'}$ in front of each term. There is no ambiguity here because the Hodge structure of each term is pure and maps are strictly compatible with these. We then compute the cohomology of this restricted double complex with respect to $\delta - \gamma$ using the spectral sequence whose E_0 -term has differential $-\gamma$ and claim that $E_2|_{k=1} = 0$. This will finish the proof of (3).

For $p' \neq q'$, $H^{p', q'}(Y_{\text{tor}}^i) = 0$ because toric varieties have no off-diagonal Hodge classes and thus $H^{p', q'}(Y^i) = H^{p', q'}(Y_{\text{ntor}}^i)$. All columns are exact at $k = 1$ by Lemma 5.8,(2). Indeed $\tilde{W}_0 \cap Y_{\text{tor}}^i = Y_{\text{ntor}}^{i+1}$ and since

$$\dim \tilde{W}_0 \cap Y_{\text{tor}}^{2\tilde{q}+1} = d - 2\tilde{q} < p + q - 2\tilde{q} = m - 1 - 2\tilde{q}$$

by $p+q > d$, the exceptional cases lie strictly below the main diagonal. We have thus shown that $E_1|_{k=1} = 0$ away from the top left corner, i.e., away from $\bigoplus_{\substack{p'+q'=m-1 \\ p' \neq q'}} H^{p', q'}(Y_{\text{ntor}}^2)$. We claim that $E_2|_{k=1} = 0$ at this term. This is equivalent to the map on the cokernels of the two top left vertical arrows induced by δ being an injection. Using the exactness of Lemma 5.8,(2), this is equivalent to the injectivity of

$$(5.9) \quad (\text{im } \gamma) \cap H_{\neq}^{m+1}(\tilde{W}_0) \xrightarrow{\delta} (\text{im } \gamma) \cap H_{\neq}^{m+1}(\tilde{W}_0 \cap Y_{\text{tor}}^1).$$

where we have used the short notation H_{\neq}^b for $\bigoplus_{p' \neq q'} H^{p', q'} H^b$. By Lemma 5.8,(1) and Poincaré duality, we have an injection

$$H_{\neq}^{m+1}(\tilde{W}_0) \xrightarrow{\delta^{\tilde{W}_0 \cap \tilde{D}}} H_{\neq}^{m+1}(\tilde{W}_0 \cap \tilde{D}^1).$$

We can't directly deduce the injectivity in (5.9) from this because $\delta = \pi_{Y_{\text{ntor}}} \circ \delta^{\tilde{W}_0 \cap \tilde{D}}$ where $\pi_{Y_{\text{ntor}}} : H_{\neq}^{m+1}(\tilde{W}_0 \cap \tilde{D}^1) \rightarrow H_{\neq}^{m+1}(\tilde{W}_0 \cap Y_{\text{tor}}^1)$ denotes the projection. We are going to show that

$$(5.10) \quad \delta^{\tilde{W}_0 \cap \tilde{D}}((\text{im } \gamma) \cap H_{\neq}^{m+1}(\tilde{W}_0)) \subseteq (\text{im } \gamma) \cap H_{\neq}^{m+1}(\tilde{W}_0 \cap Y_{\text{tor}}^1),$$

which then implies (5.9). Let us consider the diagram

$$\begin{array}{ccc} H_{\neq}^{m+1}(\tilde{W}_0) & \xrightarrow{\delta^{\tilde{W}_0 \cap \tilde{D}}} & H_{\neq}^{m+1}(\tilde{W}_0 \cap \tilde{D}^1) \\ \uparrow -\gamma^{\tilde{W}_0 \cap \tilde{D}} & & \uparrow -\gamma^{\tilde{W}_0 \cap \tilde{D}} \\ H_{\neq}^{m-1}(\tilde{W}_0 \cap \tilde{D}^1) & \xrightarrow{\delta^{\tilde{W}_0 \cap \tilde{D}}} & H_{\neq}^{m-1}(\tilde{W}_0 \cap \tilde{D}^2) \end{array}$$

It is anti-commutative because it is part of the differential in the weight spectral sequence of the punctured tubular neighbourhood of $\tilde{W}_0 \cap \tilde{D}$ in \tilde{W}_0 . Moreover, by Lemma 5.8,(2), the three terms involving the bottom and right map split as direct sums where one summand is

$$H_{\neq}^{m-1}(\tilde{W}_0 \cap Y_{\text{tor}}^1) \xrightarrow{\delta} H_{\neq}^{m-1}(\tilde{W}_0 \cap Y_{\text{tor}}^2) \xrightarrow{-\gamma} H_{\neq}^{m+1}(\tilde{W}_0 \cap Y_{\text{tor}}^1).$$

We get (5.10) from

$$\begin{aligned}
\delta^{\tilde{W}_0 \cap \tilde{D}}((\text{im } \gamma) \cap H_{\neq}^{m+1}(\tilde{W}_0)) &= (\delta^{\tilde{W}_0 \cap \tilde{D}} \circ \gamma^{\tilde{W}_0 \cap \tilde{D}})(H_{\neq}^{m-1}(\tilde{W}_0 \cap Y_{\text{tor}}^1)) \\
&= (-\gamma^{\tilde{W}_0 \cap \tilde{D}} \circ \delta^{\tilde{W}_0 \cap \tilde{D}})(H_{\neq}^{m-1}(\tilde{W}_0 \cap Y_{\text{tor}}^1)) \\
&= (-\gamma \circ \delta)(H_{\neq}^{m-1}(\tilde{W}_0 \cap Y_{\text{tor}}^1)) \\
&\subseteq (\text{im } \gamma) \cap H_{\neq}^{m+1}(\tilde{W}_0 \cap Y_{\text{tor}}^1).
\end{aligned}$$

□

5.3. The main theorem. With preparations complete, we can finish the proof of our main result by computing $h^{p,p}(\check{S}, \mathcal{F}_{\check{S}})$ as defined in (0.5), for $2p > d$.

Note that, for $2p > d$, we have by Lemma 5.5 and Prop. 5.9,(3), that

$$h^{p,p}(\mathcal{F}_{\check{S}}) = h^{p,p} \mathbb{H}^{2p}(\check{S}, \mathcal{F}_{\check{S}}) = h^{p+1,p+1} \mathbb{H}^{2p+1}(Y, \bar{A}^\bullet).$$

Proposition 5.10. *For $2p > d + 2$, we have*

- (1) $h^{p,p} \mathbb{H}^{2p-1}(Y, \bar{A}^\bullet) = h^{p,p} \mathbb{H}^{2p}(Y, \mathbb{C}) - h^{p,p} \mathbb{H}^{2p}(Y, A^\bullet).$
- (2) $\text{Gr}_i^W \mathbb{H}^m(Y, \mathbb{C}) = 0$ for $i \neq 0$ and $m > d + 2$.

Proof. We apply Gr_\bullet^W to the sequence in Thm. 4.5,(2) in order to obtain the exact sequence

$$\begin{aligned}
\cdots \rightarrow \text{Gr}_1^W \mathbb{H}^{2p-1}(Y, A^\bullet) &\rightarrow \text{Gr}_1^W \mathbb{H}^{2p-1}(Y, \bar{A}^\bullet) \rightarrow \\
\text{Gr}_0^W \mathbb{H}^{2p}(Y, \mathbb{C}) &\rightarrow \text{Gr}_0^W \mathbb{H}^{2p}(Y, A^\bullet) \rightarrow \text{Gr}_0^W \mathbb{H}^{2p}(Y, \bar{A}^\bullet) \rightarrow \cdots
\end{aligned}$$

and conclude (1) from the vanishing of the exterior terms by Prop. 5.9,(2)-(3). Similarly, replacing Gr_0^W (resp. Gr_1^W) in the above sequence by Gr_i^W (resp. Gr_{i+1}^W), we deduce (2). □

Lemma 5.11. *Let Y_{tor} denote the closure of $Y \setminus \tilde{W}_0$. We have $e^{p,q}(Y_{\text{tor}}) = 0$ for $p \neq q$ and*

$$e^{p,p}(Y_{\text{tor}}) = (-1)^{d+1-p} \sum_{\tau \in \mathcal{P}, \tau \not\subset \partial \Delta} (-1)^{\dim \tau} \binom{\dim \Delta(\tau) - \dim \tau}{p}.$$

Proof. Recall from [DK86], 2.5 that for a compact toric variety X_{Σ_0} , one has

$$h^{p,p}(X_{\Sigma_0}, \mathbb{C}) = \dim H^{2p}(X_{\Sigma_0}) = \sum_{\tau \in \Sigma_0} (-1)^{\text{codim } \tau - p} \binom{\text{codim } \tau}{p}$$

and $H^k(X_{\Sigma_0}, \mathbb{C}) = 0$ for odd k .

From the weight spectral sequence on the mixed Hodge complex computing the mixed Hodge structure on $H^\bullet(Y, \mathbb{C})$, we get

$$e^{p,q}(Y_{\text{tor}}) = \sum_{i \geq 1} (-1)^{i+1} h^{p,q}(Y_{\text{tor}}^i)$$

which is zero if $p \neq q$, so let's assume $p = q$ giving

$$\begin{aligned} e^{p,q}(Y_{\text{tor}}) &= \sum_{\omega \in \mathcal{P}_{\Delta'}} (-1)^{\dim \omega} \dim H^{2p}(X_{\omega}) \\ &= \sum_{\omega \in \mathcal{P}_{\Delta'}} (-1)^{\dim \omega} \sum_{\tau \in \mathcal{P}, \tau \supseteq \omega} (-1)^{d+1-\dim \tau - p} \binom{d+1-\dim \tau}{p}. \end{aligned}$$

Using that, for fixed τ , we have $1 = \sum_{\mathcal{P}_{\Delta'} \ni \omega \subseteq \tau} (-1)^{\dim \omega}$ and that $d+1 = \dim \Delta(\tau)$ for $\tau \in \mathcal{P} \setminus \mathcal{P}_{\partial \Delta}$ we conclude the assertion. \square

Theorem 5.12. *Given*

- *a lattice polytope Δ defining a smooth toric variety and having at least one interior lattice point;*
- *a star-like triangulation of Δ by standard simplices;*
- *Landau-Ginzburg models $w : X_{\Sigma} \rightarrow \mathbb{C}$ and $\check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$ associated to resolutions of the cone over Δ and its dual cone;*

then for the sheaves of vanishing cycles $\mathcal{F}_S = \phi_{\check{w},0} \mathbf{R}j_ \mathbb{C}_{X_{\check{\Sigma}}}[1]$ and $\mathcal{F}_{\check{S}} = \phi_{w,0} \mathbf{R}j_* \mathbb{C}_{X_{\Sigma}}[1]$ we have¹¹*

$$h^{p,q}(\mathcal{F}_S) = h^{d-p,q}(\mathcal{F}_{\check{S}})$$

giving

$$h^{p,q}(S) = h^{d-p,q}(\mathcal{F}_{\check{S}}).$$

Proof. By Example 4.3, $h^{p,q}(\mathcal{F}_S) = h^{p,q}(S)$. By Prop. 3.2,(1), we have

$$(5.11) \quad e^p(S) = h^{p,p}(S) + (-1)^d h^{p,d-p}(S),$$

while by Thm. 5.6,(3) and the vanishing by Prop. 5.9,(3), we have

$$e^p(S) = (-1)^d e^{d-p}(\check{S}, \mathcal{F}_{\check{S}}) = h^{d-p,p}(\mathcal{F}_{\check{S}}) + (-1)^d h^{d-p,d-p}(\mathcal{F}_{\check{S}}).$$

Thus it is enough to show that $h^{d-p,d-p}(\mathcal{F}_{\check{S}}) = h^{p,d-p}(S)$. This follows from (5.11) if d is even and $p = d/2$, so by the duality of Theorem 5.6,(1), it remains to show that the equality holds for $2p > d$. Using Prop. 3.2,(3), again, we just need to show for $2p > d$ that

$$(5.12) \quad h^{p,p}(\mathcal{F}_{\check{S}}) = (-1)^{d-p} \sum_{\tau \in \mathcal{P}} (-1)^{\dim \tau} \binom{\dim \Delta(\tau) - \dim \tau}{p+1}.$$

Let us assume $2p > d$. Choose $t_0 \in \mathbb{C}^*$ with $|t_0|$ small. By Prop. 5.10, we have $h^{p,p}(\mathcal{F}_{\check{S}}) = h^{p+1,p+1} H^{2p+2}(\bar{w}^{-1}(0), \mathbb{C}) - h^{p+1,p+1} H^{2p+2}(\bar{w}^{-1}(t_0), \mathbb{C})$. Note that by Prop. 5.10,(2), and the fact that $h^{p,q} H^i(Y) = 0$ for Y proper and $p+q > i$,

$$e^{p+1,p+1}(\bar{w}^{-1}(0)) = h^{p+1,p+1} H^{2p+2}(\bar{w}^{-1}(0), \mathbb{C}).$$

¹¹We use the notation $h^{p,q}(\mathcal{F}_S) = h^{p,q}(S, \mathcal{F}_S) = \sum_k h^{p,q+k} \mathbb{H}^{p+q}(S, \mathcal{F}_S)$ and likewise for \check{S} .

By the smoothness of $\bar{w}^{-1}(t_0)$, this similarly holds for $\bar{w}^{-1}(t_0)$. The contraction $\tilde{\mathbb{P}}_{\Delta} \rightarrow X_{\Sigma}$ gives an isomorphism of $\bar{w}^{-1}(t_0) \cap D$ and $\bar{w}^{-1}(0) \cap D$. Thus using Thm. 3.3,(1), we get

$$h^{p,p}(\mathcal{F}_{\tilde{S}}) = e^{p+1,p+1}(w^{-1}(0)) - e^{p+1,p+1}(w^{-1}(t_0)).$$

Moreover by a Lefschetz-type result (see [DK86], 3.9), we have Gysin isomorphisms

$$H_c^i(w^{-1}(t_0) \cap (\mathbb{C}^*)^{d+2}) \rightarrow H_c^{i+2}((\mathbb{C}^*)^{d+2}) \leftarrow H_c^i(w^{-1}(0) \cap (\mathbb{C}^*)^{d+2})$$

for $i \geq d+2$. Note that this is also true in the $\dim \Delta' = 0$ case using the fact that then $w^{-1}(0) \cap (\mathbb{C}^*)^{d+2} \cong \mathbb{C}^* \times W'$ and W' has a Newton polytope of dimension d . On the other hand $h^{p+1,p+1}H_c^i(T) = 0$ for $i < 2p+2$ and T smooth (by Poincaré duality and [PS08], Thm. 5.39), so $h^{p+1,p+1}H_c^i(w^{-1}(t) \cap (\mathbb{C}^*)^{d+2}) = 0$ for $i \leq d+2$, $t \in \{0, t_0\}$ and thus again by Thm. 3.3,(1),

$$h^{p,p}(\mathcal{F}_{\tilde{S}}) = e^{p+1,p+1}(\partial w^{-1}(0)) - e^{p+1,p+1}(\partial w^{-1}(t_0))$$

where $\partial w^{-1}(t)$ denotes the intersection of $w^{-1}(t)$ with the complement of the dense torus in X_{Σ} . Note that $Y_{\text{tor}} \subset w^{-1}(0)$, $w^{-1}(t_0) \cap Y_{\text{tor}} = \emptyset$ and the torus orbits in $X_{\Sigma} \setminus Y_{\text{tor}}$ are indexed by $\mathcal{P}_{\partial\Delta}$. Decomposing in torus orbits yields

$$h^{p,p}(\mathcal{F}_{\tilde{S}}) = e^{\hat{p},\hat{p}}(Y_{\text{tor}}) - \sum_{\tau \in \mathcal{P}_{\partial\Delta}} (e^{\hat{p},\hat{p}}(w^{-1}(t_0) \cap T_{\tau}) - e^{\hat{p},\hat{p}}(w^{-1}(0) \cap T_{\tau}))$$

where $\hat{p} = p+1$. By Cor. 1.22, for $\tau \in \mathcal{P}_{\partial\Delta}$, we have

$$\begin{aligned} w^{-1}(t_0) \cap T_{\tau} &\cong H^{\text{codim } \mathcal{P}_*(\tau)-1} \times (\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau)-\dim \tau}, \\ w^{-1}(0) \cap T_{\tau} &\cong H^{\text{codim } \mathcal{P}_*(\tau)-2} \times (\mathbb{C}^*)^{\dim \mathcal{P}_*(\tau)-\dim \tau+1} \end{aligned}$$

Note that, for $\tau \in \mathcal{P}_{\partial\Delta}$, $w^{-1}(t_0) \cap T_{\tau}$ is non-empty iff $\text{codim } \mathcal{P}_*(\tau) \geq 1$ whereas $w^{-1}(0) \cap T_{\tau}$ is non-empty iff $\text{codim } \mathcal{P}_*(\tau) \geq 2$; moreover, $\mathcal{P}_*(\tau) = \Delta(\tau)$. The assertion now follows from Lemma 5.11 and Lemma 3.4. \square

6. COMPLEMENTS

6.1. Relation to discrete Legendre transforms and toric degenerations. Recall from [GS03] the definition of a polarized tropical manifold, $(B, \mathcal{P}, \varphi)$, where B is an integral affine manifold with singularities, \mathcal{P} a polyhedral decomposition of B into lattice polyhedra, and φ a strictly convex multi-valued piecewise linear function with integral slopes on B . A tropical manifold may have a boundary as well as unbounded cells. Here, we will only need tropical manifolds without singularities, isomorphic to polyhedra, and with φ single-valued. Also recall from [GS03] how to associate a polarized tropical manifold to a polarized toric degeneration of an algebraic variety. This polarized tropical manifold is the dual intersection complex of the toric degeneration. In addition, recall the notion of the discrete Legendre transform. The latter associates another polarized tropical manifold

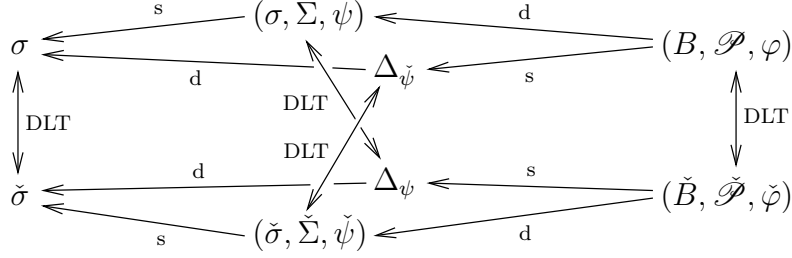


FIGURE 12. Tropical manifolds and their relationships: DLT marks a discrete Legendre transform, s marks a subdivision, d marks a degeneration

$(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ to $(B, \mathcal{P}, \varphi)$ which in turn transforms back to $(B, \mathcal{P}, \varphi)$ upon another application of a discrete Legendre transform. This transform has been found to realize mirror symmetry for maximally unipotent toric degenerations. We now relate this to our mirror construction.

We return to the situation of (0.1), (0.2), so we are given dual cones $\sigma, \check{\sigma}$ subdivided into fans $\Sigma, \check{\Sigma}$ giving Landau-Ginzburg models $w : X_{\Sigma} \rightarrow \mathbb{C}, \check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$. We shall build toric degenerations of $X_{\Sigma}, X_{\check{\Sigma}}$ whose corresponding dual intersection complexes $(B, \mathcal{P}, \varphi)$ and $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ are related by discrete Legendre transform. This will show that the mirror symmetry of (0.1), (0.2) fits into the general setup of the Gross-Siebert program.

We use a standard method of building toric degenerations, see e.g., [NS06]. Let $B \subseteq M_{\mathbb{R}}$ be a (non-compact) polyhedron with a polyhedral decomposition \mathcal{P} and a strictly convex piecewise linear function φ with integral slopes. Let $\check{\Sigma}$ be the fan in $\bar{M}_{\mathbb{R}} = M_{\mathbb{R}} \oplus \mathbb{R}$ defined by

$$\check{\Sigma} = \bigcup_{\tau \in \mathcal{P}} \{\text{faces of Cone}(\tau)\},$$

where now one must take care to take the closure in defining

$$\text{Cone}(\tau) = \overline{\{(rm, r) \mid m \in \tau, r \geq 0\}},$$

as τ need not be compact. We say (B, \mathcal{P}) has *asymptotic fan* Σ if

$$\Sigma = \{\tau \in \check{\Sigma} \mid \tau \subseteq M_{\mathbb{R}} \times \{0\}\}.$$

If (B, \mathcal{P}) has asymptotic fan Σ , then the projection $\bar{M}_{\mathbb{R}} \rightarrow \mathbb{R}$ induces a map of toric varieties $X_{\check{\Sigma}} \rightarrow \mathbb{A}^1$ whose general fibre is isomorphic to X_{Σ} . This is a toric degeneration of X_{Σ} .

Further, φ induces a piecewise linear function $\check{\varphi}$ on $\check{\Sigma}$ as the unique piecewise linear extension of φ to $\check{\Sigma}$, thinking of φ as a function on $B \times \{1\}$.

We then have:

Theorem 6.1. *Let $w : X_{\Sigma} \rightarrow \mathbb{C}, \check{w} : X_{\check{\Sigma}} \rightarrow \mathbb{C}$ be dual Landau-Ginzburg models as given in (0.1), (0.2). Let ψ (resp. $\check{\psi}$) denote the piecewise linear function inducing the subdivision*

Σ of σ (resp. $\tilde{\Sigma}$ of $\tilde{\sigma}$). Then there are polarized tropical manifolds $(B, \mathcal{P}, \varphi)$, $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$, related by discrete Legendre transform, yielding fans $\tilde{\Sigma}, \check{\tilde{\Sigma}}$ as above, such that

- Σ and $\tilde{\Sigma}$ are the asymptotic fans of (B, \mathcal{P}) and $(\check{B}, \check{\mathcal{P}})$ respectively.
- Thinking of Σ and $\tilde{\Sigma}$ as subfans of $\check{\tilde{\Sigma}}$ and $\tilde{\Sigma}$ respectively via $\tau \mapsto \tau \times \{0\}$, we have $\tilde{\varphi}|_{\Sigma} = \psi$ and $\check{\tilde{\varphi}}|_{\tilde{\Sigma}} = \check{\psi}$.

Proof. By adding a linear function, we may assume that $\check{\psi}|_{\check{\tau}} \equiv 0$ for some maximal cone $\check{\tau} \in \check{\tilde{\Sigma}}$. Let $\check{B} = \Delta_{\check{\psi}}$ be the Newton polyhedron of $\check{\psi}$ and $\Delta_{\check{\psi}}^c$ the convex hull of its compact faces. We have $\Delta_{\check{\psi}} = \Delta_{\check{\psi}}^c + \check{\sigma}$. Then

$$\text{Cone}(\check{B}) = \bigcup_{t \geq 0} (t\Delta_{\check{\psi}}^c + \check{\sigma}, t).$$

Let $\check{\psi}^c$ be the zero function on $\Delta_{\check{\psi}}^c$. For $(b, t) \in \text{Cone}(\check{B})$ we set

$$\check{\varphi}(b) = \min\{\check{\psi}^c(p) + \check{\psi}(q) \mid p + q = b, p \in t\Delta_{\check{\psi}}^c, q \in \check{\sigma}\}.$$

This function is piecewise affine and convex on $\text{Cone}(\check{B})$. Let $\check{\tilde{\Sigma}}$ be the fan of maximal domains of linearity of $\check{\varphi}$, let $\check{\mathcal{P}}$ be the induced cell decomposition on \check{B} , identified with $\check{B} \times \{1\}$, and let $\check{\varphi} = \check{\varphi}|_{\check{B} \times \{1\}}$. Note that $\check{\varphi}|_{\check{\sigma} \times \{0\}} = \check{\psi}$ by construction, so $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ has the desired properties.

To get the dual, set

$$\check{P} = \{(b, s) \in \check{B} \times \mathbb{R} \mid s \geq \check{\varphi}(b)\},$$

and let $\tilde{\Sigma}$ be the normal fan to \check{P} , with piecewise linear function $\tilde{\varphi}$ induced by \check{P} . Then $\tilde{\Sigma}$ has support $|\tilde{\Sigma}|$ contained in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$, and if we take $B = |\tilde{\Sigma}| \cap (N_{\mathbb{R}} \times \{1\})$, then $|\tilde{\Sigma}| = \text{Cone}(B)$. Furthermore, B inherits a decomposition \mathcal{P} from $\tilde{\Sigma}$, and $\tilde{\Sigma}$ is obtained by taking cones over elements of \mathcal{P} . The asymptotic fan of B is the normal fan to \check{B} , i.e., Σ , and $\tilde{\varphi}|_{\Sigma} = \psi$, since \check{B} was the Newton polytope of $\check{\psi}$. Setting $\varphi = \tilde{\varphi}|_{B \times \{1\}}$, one checks that $(B, \mathcal{P}, \varphi)$ is the discrete Legendre transform of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$. \square

We note that the choice of $\mathcal{P}, \check{\mathcal{P}}$ given in the proof is not canonical: there may be many choices.

Given the pair $\tilde{\Sigma}, \check{\tilde{\Sigma}}$ produced by the theorem, we can construct Landau-Ginzburg models $\tilde{w}, \check{\tilde{w}}$ on the families $X_{\tilde{\Sigma}}, X_{\check{\tilde{\Sigma}}} \rightarrow \mathbb{A}^1$. Let

$$\begin{aligned} \tilde{w} &:= \sum_{\check{\rho}} c_{\check{\rho}} z^{(n_{\check{\rho}}, \check{\psi}(n_{\check{\rho}}))} \\ \check{\tilde{w}} &:= \sum_{\rho} c_{\rho} z^{(m_{\rho}, \psi(m_{\rho}))} \end{aligned}$$

where the sums are over all one-dimensional rays $\check{\rho}$ in $\check{\tilde{\Sigma}}$ (ρ in Σ), with $n_{\check{\rho}} (m_{\rho})$ the primitive generator of $\check{\rho} (\rho)$. One checks that these are regular functions on $X_{\tilde{\Sigma}}$ and $X_{\check{\tilde{\Sigma}}}$ respectively. Note that by identifying t with $z^{(0,1)}$, away from the fibre of $X_{\tilde{\Sigma}} \rightarrow \mathbb{A}^1$ over 0, we can view

\tilde{w} as giving a family w_t of Landau-Ginzburg potentials on X_Σ , parameterized by t , with $w_t = \sum_{\tilde{\rho}} c_{\tilde{\rho}} t^{\tilde{\psi}(n_{\tilde{\rho}})} z^{n_{\tilde{\rho}}}$.

Examples 6.2. For a suitable choice of $\tilde{\psi}$, applying this construction to $(\check{\sigma}, \check{\Sigma}, \check{\psi})$ yields the degeneration given in Rem. 1.25. Applying this to (σ, Σ_*, h_*) yields the one given in Rem 1.9 possibly up to a change of the coefficients.

6.2. Relation to conic bundles and the work of others. Mirror symmetry can be studied locally by looking at a conic bundle of the shape

$$(6.1) \quad uv = f(w_1, \dots, w_n)t$$

in $\mathbb{A}^2 \times (\mathbb{C}^*)^n \times \mathbb{A}^1$ where u, v, w_1, \dots, w_n, t are coordinates of the factors in the given order. Here, t is a family parameter. One should think of $t = 0$ as being a toric degeneration of a non-compact Calabi-Yau manifold given by a general fibre for $t \neq 0$ (assuming f defines a smooth subvariety of $(\mathbb{C}^*)^n$). For a fixed $t \neq 0$, the projection to $(\mathbb{C}^*)^n$ yields a conic bundle with discriminant $f = 0$ which is also the singular locus of the total space of the family. This local Calabi-Yau is an essential building block of the toric degenerations studied by Gross-Siebert in [GS03].

In [AAK], the authors work out an understanding of mirror symmetry for varieties of general type from the point of view of Strominger-Yau-Zaslow torus fibrations and blow-ups. Their basic setup is a conic bundle as given above. It arises from when one blows up $\mathbb{A}^1 \times (\mathbb{C}^*)^n$ in $0 \times \{f = 0\}$.

Let us understand how a conic bundle appears in our construction: In Prop. 1.8, we have compactified $X_{\check{\Sigma}}$ to $X_{\check{\Sigma}}$, a \mathbb{P}^1 -bundle over \mathbb{P}_Δ . To turn the resulting rational map

$$\tilde{w} : X_{\check{\Sigma}} \dashrightarrow \mathbb{P}^1$$

into a regular one, we have blown up the intersection of $\tilde{w}^{-1}(0)$ with the divisor at infinity. In a neighbourhood of the center of the blow-up, we have precisely the setup of [AAK]: Indeed, recall from (1.3) that

$$\tilde{w} = \sum_{m \in \Delta \cap M} c_m z^{(m,1)}.$$

We restrict this to $(\mathbb{C}^*)^{d+2}$, then compactify to the \mathbb{P}^1 -bundle. The graph of \tilde{w} is given by

$$v_1 u_0 f - u_1 v_0,$$

setting $f = \sum_{m \in \Delta \cap M} c_m z^{(m,0)}$, $u_1 = z^{(0,-1)}$, $u_0 = z^{(0,1)}$, v_0, v_1 being homogeneous coordinates on the target \mathbb{P}^1 of \tilde{w} . Most importantly, in the neighbourhood at infinity given by setting $u_0 = 1$, we blow up the locus $f = u_1 = 0$ just as in [AAK].

We have thus seen that in the resolution $\tilde{w} : \tilde{X}_{\check{\Sigma}} \rightarrow \mathbb{P}^1$, there is no critical value in \mathbb{C}^* and the fibres over 0 and ∞ have isomorphic singular locus with trivial monodromy on cohomology. Our construction sits at 0 whereas the conic bundle is a neighbourhood of ∞ .

The potentials we gave in (1.2) and (1.3) are not quite the right ones as expected from the SYZ mirror symmetry construction in [Aur07]. Roughly speaking, Auroux’s construction associates to a manifold X with effective anticanonical divisor D a mirror as a manifold \check{X} with a potential $\check{w} : \check{X} \rightarrow \mathbb{C}$ constructed from counting certain Maslov index two holomorphic disks in X . In our setup where $X = X_\Sigma$ (resp. $X = X_{\check{\Sigma}}$), we implicitly use the toric boundary divisor as a choice for D . The naive potential we are using — up to changing its coefficients — only counts a subset of all holomorphic disks. This is because our potential can be viewed as given by a count of *tropical* disks, see [Gr10]. Tropical geometry, however, cannot see (degenerate) disks with components mapping into D . There are typically algebraic curves contained in D with non-negative Chern number which can be glued to disks visible tropically to obtain additional Maslov index two disks contributing to the potential. These can contribute additional monomials.

There doesn’t seem to be an easy way to describe all the curves that contribute additionally to the potential directly from Σ or $\check{\Sigma}$. However, [CPS11] provides an approach for obtaining what should be the correct potential in the context of the Gross-Siebert program. The authors choose a smoothing of D which is reflected in the tropical manifold (see §6.1) by “pulling in singularities from infinity” such that all unbounded rays become parallel, i.e., making the boundary in the discrete Legendre dual totally geodesic. This requires introducing singularities in the affine manifolds introduced in §6.1. Once this is done, the techniques of [GS11] can be applied to obtain a tropical description of what should be interpreted as Maslov index zero disks. Finally, [CPS11] then demonstrates how to construct a well-defined potential from this data. Presumably, this will give the same potential as the one obtained in [AAK].

It can be checked in examples that carrying this out in our setup does change the potentials but does not affect its critical locus and sheaf of vanishing cycles. This point of view will be explored in more detail elsewhere.

6.3. Singular fibres and deformations of the potential. In the study of mirror symmetry involving any Landau-Ginzburg model, there is always a question as to which singular fibres should contribute. Except in the case of the mirror of the cubic three-fold, we only make use of the zero fibre, whereas in the case of the cubic three-fold (see §7), we in fact make use of all singular fibres. This raises the question of justifying these choices.

We believe that these choices can be justified mathematically by incorporating the Landau-Ginzburg picture into the Gross-Siebert mirror symmetry program as discussed in §6.1. If t is the parameter for the family, with $t = 0$ the degenerate fibre, and w_t the t -dependent potential, then one can explore what happens to the critical values of w_t as $t \rightarrow 0$. Those critical values which go to ∞ as $t \rightarrow 0$ are the ones which should be ignored. This is essentially the behaviour already observed in [FOOO] in the case of mirrors of toric

varieties. There, the authors work over the Novikov ring which implicitly disregards the unwanted critical values.

To see this in practice in several of the explicit examples of this paper, consider first the example of the genus two curve. It is easiest to describe a natural toric degeneration in terms of the compactification described in Example 1.6, where one takes a degeneration of the form $xy - tz^2 = uv - tz^3 = 0$, while we take $w = c_x x + c_y y + c_z z + c_u u + c_v v$ as before. One finds that 0 is always a critical value, but the remaining critical values behave like order $t^{-1/2}$, and hence go to infinity as $t \rightarrow 0$.

On the other hand, a natural degeneration for the mirror of the cubic three-fold is given by $x_0 x_4 u_1 u_2 u_3 = ts^3$, $u_1 + u_2 + u_3 = s$, and one checks the critical values are 0 and $\pm 6\sqrt{3t}$, which do not go to infinity.

We will not be more precise here, as this will be explored in greater detail elsewhere.

6.4. Complete intersections in toric varieties. A Landau-Ginzburg model for a complete intersection in a toric variety was already given in [HW09] based on [BB94]. It closely relates to the local models of the logarithmic singularities given in [Rud10] based on [GS10]. Let \mathbb{P}_Δ be a smooth projective toric variety, D_1, \dots, D_k effective toric divisors with Newton polytopes $\Delta_1, \dots, \Delta_k$ and non-degenerate global sections f_1, \dots, f_k of the corresponding line bundles. We require f_1, \dots, f_k to be transversal, i.e., $(\partial_{x_j} f_i)$ has rank k at each point of \mathbb{P}_Δ , where x_j are local coordinates on \mathbb{P}_Δ and the f_i are viewed as regular functions using a local trivialisation of $\mathcal{O}(D_i)$. Transversality of f_1, \dots, f_k is implied if $\Delta_1, \dots, \Delta_k$ are transversal, i.e., their tangent spaces embed as a direct sum in $M_\mathbb{R}$. We define the cone

$$\sigma = \text{Cone}(\text{Conv}(\Delta_1 \times \{e_1\}, \dots, \Delta_k \times \{e_k\}))$$

in $M_\mathbb{R} \oplus \mathbb{R}^k$ where e_1, \dots, e_k is the standard basis of \mathbb{R}^k . Its dual cone is given by

$$\check{\sigma} = \{(n, a_1, \dots, a_k) \mid a_i \geq \varphi_{\Delta_i}(n)\} \subseteq N_\mathbb{R} \oplus \mathbb{R}^k.$$

Let $\check{\Sigma}$ denote the star subdivision of $\check{\sigma}$ along the cone generated by e_1^*, \dots, e_k^* . It is not hard to see that $X_{\check{\Sigma}} = \text{Tot}(\mathcal{O}_{\mathbb{P}_\Delta}(-D_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}_\Delta}(-D_k))$. Setting $u_i = z^{e_i}$, we find that

$$\check{w} = \sum_i u_i f_i$$

is a regular function on $X_{\check{\sigma}} = \text{Spec}[\sigma \cap (M \oplus \mathbb{Z}^k)]$ with Newton polytope $\hat{\Delta} = \text{Conv}(\Delta_1 \times \{e_1\}, \dots, \Delta_k \times \{e_k\})$. We pull \check{w} back to $X_{\check{\Sigma}}$. The smoothness of

$$S = \text{crit}(\check{w}) = V(f_1) \cap \dots \cap V(f_k)$$

follows from the transversality of the f_i . We construct the mirror of S as follows. Let Σ_* be the star subdivision of σ along the subcone generated by

$$\hat{\Delta}' = \text{Conv}\{\Delta'_1 \times \{e_1\}, \dots, \Delta'_k \times \{e_k\}\}$$

where Δ'_i denotes the convex hull of the interior lattice points of Δ_i . Let Σ be a refinement of Σ_* given by a triangulation of $\hat{\Delta}$ such that each cone in Σ is a standard cone. Since this does not need to exist, more generally one needs to allow simplicial cones, see §6.5. However, with the assumption made, X_Σ is smooth. Moreover, \check{w} is now in the shape of (0.2). We define the potential w on X_Σ as in (0.1) and take the pair

$$(\check{S} = \text{Sing}(w^{-1}(0)), \mathcal{F}_{\check{S}} = \phi_{w,0} \mathbb{C}_{X_\Sigma}[1])$$

for the mirror dual of S . One can show that $\dim \check{S} = \dim S$, so that $(\check{S}, \mathcal{F}_{\check{S}})$ is plausible as a mirror of S , in analogy with the hypersurface case.

6.5. A refinement of the general conjecture using orbifolds. We state here a refined version of the conjecture concerning Landau-Ginzburg models defined using dual cones σ and $\check{\sigma}$ of the statement made in the introduction. Given a cone $\sigma \subseteq M$, one can define a fan Σ_* refining σ in a canonical way, by taking Σ_* to be the cones over faces of the convex hull σ° of the set of points $\sigma \cap (M \setminus \{0\})$. The corresponding toric variety X_{Σ_*} is not necessarily a resolution of X_σ , however it is always Gorenstein. One can subdivide each bounded face of σ° into elementary simplices, i.e., simplices which do not contain any integral points of M other than vertices. This refinement Σ , which is not unique, yields an orbifold resolution $X_\Sigma \rightarrow X_\sigma$ which is crepant over X_{Σ_*} . We can follow the same procedure for $\check{\sigma}$, hence obtain as in the introduction Landau-Ginzburg potentials

$$\begin{aligned} w : X_\Sigma &\rightarrow \mathbb{C} \\ \check{w} : X_{\check{\Sigma}} &\rightarrow \mathbb{C}. \end{aligned}$$

We pose the following

Conjecture 6.3. *There is a version of the sheaf of vanishing cycles for orbifolds, where each $H^{p,q}(Y^j)$ in Lemma 4.7, (3) is replaced by $H_{\text{orb}}^{p,q}(Y^j)$. Defining $h_{\text{orb}}^{p,q}(X_\Sigma, w)$ and $h_{\text{orb}}^{p,q}(X_{\check{\Sigma}}, \check{w})$ then analogously to Cor. 0.3, $n = \dim X_\Sigma$, we have*

$$h_{\text{orb}}^{p,q}(X_\Sigma, w) = h_{\text{orb}}^{n-p,q}(X_{\check{\Sigma}}, \check{w}).$$

Assuming a renormalization flow argument works in the orbifold case, the last statement of the conjecture holds true in the Calabi-Yau case as was shown in [BB96].

Note that in the particular case of this paper, where σ is the cone over a polytope, the resolutions we use are special cases of the above resolutions. We believe, based on this and some other examples, that these special types of resolutions allow us to make the above statement using just the critical value 0 on both sides. This holds for the case considered in this paper. On the other hand, using arbitrary total resolutions as in (0.1), (0.2) in some sense adds geometry that wasn't originally there.

The simplest case of this conjecture which is not a Calabi-Yau situation and not already verified in this paper would be where σ and $\check{\sigma}$ are both two-dimensional cones defining

non-Gorenstein rational quotient singularities. We have verified the conjecture in several such explicit examples.

7. THE CUBIC THREE-FOLD

This section can be viewed as being complementary to the main discussion of this paper. We shall consider one example of a mirror to a non-trivial Fano threefold, namely the cubic hypersurface in \mathbb{P}^4 . In the general type case considered in the bulk of the paper, a hypersurface gave rise to a Landau-Ginzburg mirror whose dimension is two more than that of the starting hypersurface. We then obtained a mirror of the correct dimension by passing to the critical locus of one fibre of the potential. In the case of a Fano hypersurface, we can't do this. The critical locus has a very different character, and it doesn't make sense to restrict to this critical locus.

There are then several alternative approaches one can take. First, there are already a number of constructions of Landau-Ginzburg mirrors to Fano hypersurfaces in the literature, giving mirrors of the correct dimension. Second, we can use a different technique to reduce the dimension, namely Knörrer periodicity, see §2.1. We examine the various approaches:

(1) In [Gi96], A. Givental proposed a mirror given as the pair (X, w) where X is defined by the equations $x_0x_4u_1u_2u_3 = 1$, $u_1 + u_2 + u_3 = 1$ in $(\mathbb{C}^*)^5$. Here x_0, x_4, u_1, u_2, u_3 are coordinates on this algebraic torus, and $w = x_0 + x_4$.

It will be helpful to partially compactify this as follows, replacing X by the subvariety of $\mathbb{A}^2 \times \mathbb{P}^3$ defined by $x_0x_4u_1u_2u_3 = s^3$, $u_1 + u_2 + u_3 = s$, where now x_0, x_4 are coordinates on \mathbb{A}^2 and u_1, u_2, u_3, s are coordinates on \mathbb{P}^3 .

(2) In [ILP11], the authors proposed a “weak Landau-Ginzburg model” mirror for the cubic threefold, the function $\frac{(x+y+1)^3}{xyz} + z$ on the algebraic torus $(\mathbb{C}^*)^3$. This is in fact the same as w in construction (1) after a change of coordinates: take $x = u_1/u_3$, $y = u_2/u_3$, $z = x_4$, and note that in (1) we have the equation $x_0x_4u_1u_2u_3 = (u_1 + u_2 + u_3)^3$. Then $(x + y + 1)^3/(xyz) = (u_1 + u_2 + u_3)^3/(u_1u_2u_3x_4) = x_0$, so in fact $w = x_0 + x_4$.

(3) The construction of this paper proposes a five-dimensional Landau-Ginzburg mirror to the cubic threefold. One begins with a polytope Δ in \mathbb{R}^4 which is the standard simplex rescaled by a factor of 3. The cone σ over Δ in \mathbb{R}^5 is generated by

$$v_0 = (0, 0, 0, 0, 1), v_1 = (3, 0, 0, 0, 1), \dots, v_4 = (0, 0, 0, 3, 1).$$

The dual cone $\check{\sigma}$ is generated by

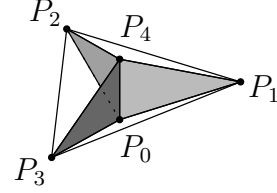
$$(7.1) \quad (1, 0, 0, 0, 0), \dots, (0, 0, 0, 1, 0), (-1, -1, -1, -1, 3).$$

The corresponding toric variety is desingularized by subdividing along the ray generated by $(0, 0, 0, 0, 1)$, and the argument of Prop. 1.8,(3) and Example 4.3 shows that the Landau-Ginzburg model on this toric variety corresponds to a cubic three-fold. Dually, the toric

variety X_σ carries a potential $w = s_0 + \cdots + s_4 - s_5$, where s_0, \dots, s_4 correspond to the vectors of the list (7.1) and s_5 to $(0, 0, 0, 0, 1)$. We note the choice of sign in front of s_5 is arbitrary, as in theory we could use any coefficients. This choice fits better with the previous models. To see the relationship between this five-dimensional model and the mirror cubic of (1) or (2), we proceed as follows.

Take a partial crepant resolution of X_σ by taking a subdivision \mathcal{P}_* of Δ via a star subdivision at $v = (1, 1, 1, 0, 1) = (v_1 + v_2 + v_3)/3$. Thus the edges in this star subdivision with endpoint v have as other endpoint v_0, v_1, v_2, v_3 and v_4 . This gives a fan Σ_* refining σ . The exceptional divisor E of the partial resolution corresponding to the vertex v is then described by a quotient fan Σ_v in \mathbb{R}^4 with one-dimensional cones generated by the vectors in the left column of the following table (eliminating the last coordinate):

	x_0	u_1	u_2	u_3	x_4	s
$P_0 = (-1, -1, -1, 0)$	3					1
$P_1 = (2, -1, -1, 0)$		3				1
$P_2 = (-1, 2, -1, 0)$			3			1
$P_3 = (-1, -1, 2, 0)$				3		1
$P_4 = (-1, -1, -1, 3)$					3	1



The remainder of the table displays the vanishing order of certain sections of the anti-canonical bundle of X_{Σ_v} and regular functions (to be explained shortly) on the divisors corresponding to P_0, \dots, P_4 . The diagram on the right shows the combinatorics of the fan Σ_v . It is in fact an incomplete fan with the three four-dimensional cones

$$\langle P_0, P_1, P_2, P_4 \rangle, \quad \langle P_0, P_1, P_3, P_4 \rangle, \quad \langle P_0, P_2, P_3, P_4 \rangle.$$

We can describe the corresponding toric variety $E \cong X_{\Sigma_v}$ as follows. Consider the Newton polyhedron for $-K_E$. This divisor is represented by the piecewise linear function which takes the value 1 on each P_i . Note that $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 0) \in N$ all lie in the Newton polyhedron for this piecewise linear function, and hence represent sections of $-K_E$, which we write as u_1, u_2, u_3 and s respectively. On the other hand, the monomials $x_0 = z^{(-1, -1, -1, -1)}, x_4 = z^{(0, 0, 0, 1)}$ are in fact regular functions on E . Using $x_0, x_4, u_1, u_2, u_3, s$ to map X_{Σ_v} to $\mathbb{A}^2 \times \mathbb{P}^3$, one sees one has the relation $x_0 x_4 u_1 u_2 u_3 = s^3$. One checks easily that this map is in fact an embedding. In particular, we can view E as a partial compactification of the torus $x_0 u_1 u_2 u_3 x_4 = 1$ appearing in Givental's construction (1).

As the notation suggests, the partial resolution coming from the decomposition \mathcal{P}_* can be viewed as analogous to the partial resolution we used in §1 with the same notation. This latter decomposition was completely canonical, whereas in the Fano case, there is no such canonical resolution.

To relate the five-dimensional Landau-Ginzburg model on X_{Σ_*} to a three-dimensional one, restrict w to any affine subset corresponding to a maximal cone of Σ_* containing v , say the cone spanned by v, v_0, v_1, v_2, v_4 . The dual cone is spanned by $(0, 0, 1, 0, 0)$, $(-1, -1, -1, -1, 3)$, $(1, 0, -1, 0, 0)$, $(0, 1, -1, 0, 0)$, and $(0, 0, 0, 1, 0)$. In addition, the dual cone contains the integral point $(0, 0, -1, 0, 1)$. In fact, these six integral points are generators of the monoid of integral points of the dual cone. If we take monomial functions y, x_0, u_1, u_2, x_4, s corresponding to these six integral points, we note that we have the relation $x_0 x_4 u_1 u_2 = s^3$. Furthermore, we can write the potential

$$s_0 + s_1 + s_2 + s_3 + s_4 - s_5 = yu_1 + yu_2 + y + x_4 + x_0 - ys = y(1 - s + u_1 + u_2) + x_0 + x_4.$$

By Knörrer periodicity (see Prop. 2.1), the associated category is equivalent to the category of the LG model on the locus $y = u_1 + u_2 + 1 - s = 0$ with potential $x_0 + x_4$. Note that $y = 0$ gives an affine piece of the exceptional divisor E . Checking this description on each affine subset corresponding a maximal cone, one finds our five-dimensional LG model should be equivalent to the three-dimensional one given on the three-fold $V(u_1 + u_2 + u_3 - s) \subseteq E \subseteq \mathbb{A}^2 \times \mathbb{P}^3$ with $w = x_0 + x_4$. Restricting this potential to the intersection of this three-fold with the big torus orbit on E gives Givental's model in (1). Furthermore, we have now obtained the partial compactification of Givental's model described there, hence justifying this choice of partial compactification. Thus we will take as a starting point the model

$$(X = V(x_0 x_4 u_1 u_2 u_3 - s^3, u_1 + u_2 + u_3 - s) \subseteq \mathbb{A}^2 \times \mathbb{P}^3, w = x_0 + x_4).$$

In fact, (X, w) is not quite what we want for the mirror: we should take a crepant resolution of X . Once this is done, we can describe the sheaf of vanishing cycles and compute its cohomology.

Given the fan Σ_v describing the exceptional divisor E above, we note that the toric divisor of E corresponding to the ray generated by P_i is

$$(7.2) \quad \begin{aligned} s = x_i = 0 & \quad \text{if } i \in \{0, 4\}, \\ s = u_i = 0 & \quad \text{if } i \in \{1, 2, 3\}. \end{aligned}$$

We can now refine the fan Σ_v to resolve the toric singularities. Note that the hyperplane $u_1 + u_2 + u_3 = s$ defining X inside X_{Σ_v} is Σ_v -regular. In particular, X is disjoint from zero-dimensional toric strata of X_{Σ_v} . So to resolve X , we just need to choose a subdivision of Σ_v which resolves X_{Σ_v} away from the zero-dimensional strata. Hence, we do not need to specify the subdivision of the four-dimensional cones of X_{Σ_v} ; rather, we will just subdivide the three-dimensional cones. In addition, the hyperplane $u_1 + u_2 + u_3 = s$ is disjoint from the one-dimensional toric strata corresponding to the three-dimensional cones generated by P_0, P_i, P_j or P_4, P_i, P_j with $i, j \in \{1, 2, 3\}$ as can be seen from (7.2). Therefore, we only need to specify subdivisions of the remaining three three-dimensional cones which we

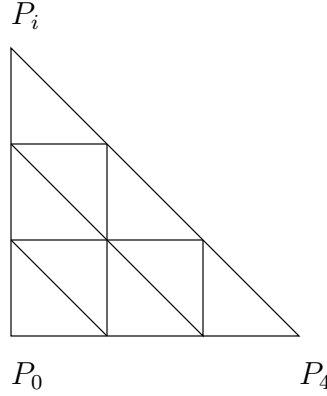


FIGURE 13. The subdivision of three-dimensional cones in Σ_v is induced by the above triangulation of the triangle with vertices P_0, P_i and P_4 , as depicted.

do as given in Figure 13. Any choice extending the subdivision of these three cones to a subdivision Σ of Σ_v will not affect the resolution of X .

Now let \tilde{X} be the proper transform of X under the blow-up $\pi : X_\Sigma \rightarrow X_{\Sigma_v}$, and write $\tilde{w} : \tilde{X} \rightarrow \mathbb{C}$ for the composition $w \circ \pi$. The smoothness of \tilde{X} follows because it is Σ -regular and X_Σ is smooth outside of zero-dimensional strata. The crepancy of $\tilde{X} \rightarrow X$ follows from that of $X_\Sigma \rightarrow X_{\Sigma_v}$ by the adjunction formula. There is an operation of S_3 on the open subvariety of X_Σ given by the union of all torus orbits that intersect \tilde{X} . It is given on the fan by the permutation of the first three coordinates and lifts to \tilde{X} .

The following summarizes the relevant geometry.

Lemma 7.1. *$\tilde{w}^{-1}(0)$ has six irreducible components, each with multiplicity one, which we shall write as D_1, D_2, D_3, S_1, S_2 and W_0 . Here*

- D_i is a del Pezzo surface of degree 6, and is the intersection of the toric divisor corresponding to the ray generated by $(P_0 + P_4 + P_i)/3$ with \tilde{X} .
- S_i is a rational scroll blown up in three points, and is the intersection of the toric divisor corresponding to the ray generated by $(iP_0 + (3-i)P_4)/3$ with \tilde{X} .
- W_0 is the proper transform of $w^{-1}(0)$, and is a non-singular quasi-projective variety.

Furthermore, these irreducible components intersect each other pairwise transversally, as follows:

- $\ell := S_1 \cap S_2 = W_0 \cap S_1 = W_0 \cap S_2$ is isomorphic to \mathbb{P}^1 .
- $Q_i := D_i \cap \ell$ is a point.
- $D_i \cap S_j \cong \mathbb{P}^1$ for each i, j .
- $D_i \cap W_0 \cong \mathbb{P}^1$ for each i .
- $D_i \cap D_j = \emptyset$ for $i \neq j$.

Thus in particular general points of ℓ are triple points of $\tilde{w}^{-1}(0)$.

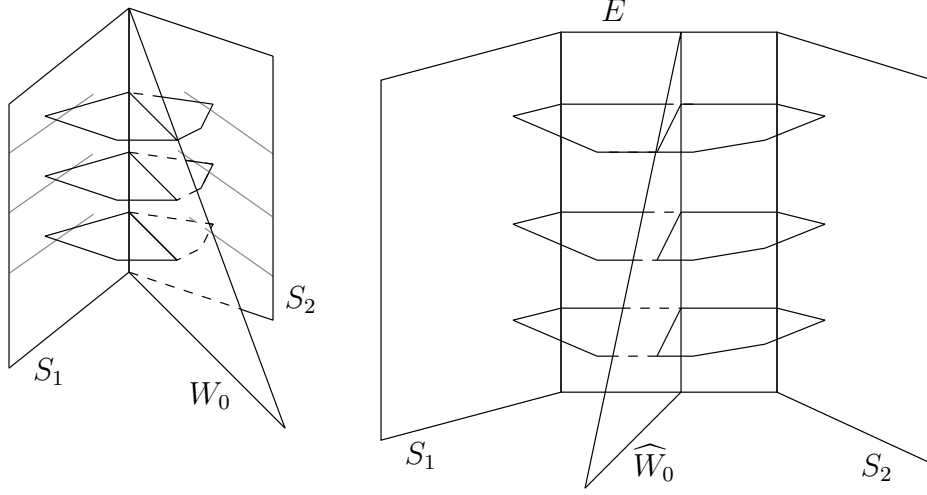


FIGURE 14. $\tilde{w}^{-1}(0)$ is depicted on the left and $\hat{w}^{-1}(0)$ is depicted on the right. The grey lines on the left show components of the singular fibres of the scrolls. The unlabelled horizontal components on the left are D_1, D_2, D_3 and on the right are $\hat{D}_1, \hat{D}_2, \hat{D}_3$.

Let $\hat{\pi} : \hat{X} \rightarrow \tilde{X}$ be the blow-up of ℓ , $\hat{w} = \tilde{w} \circ \hat{\pi}$. Then $\hat{w}^{-1}(0)$ is normal crossings, with irreducible components \hat{D}_i , $i = 1, 2, 3$, S_j , $j = 1, 2$, \hat{W}_0 , and E . Here

- $\hat{\pi} : S_j \rightarrow S_j$ and $\hat{\pi} : \hat{W}_0 \rightarrow W_0$ are isomorphisms.
- $\hat{\pi} : \hat{D}_i \rightarrow D_i$ is the blow-up of D_i at the point $S_1 \cap S_2 \cap D_i$.
- E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and appears with multiplicity three in $\hat{w}^{-1}(0)$.

These components intersect as follows:

- $S_1 \cap S_2 = \emptyset$, $\hat{D}_i \cap \hat{D}_j = \emptyset$ for $i \neq j$, $\hat{W}_0 \cap S_i = \emptyset$.
- $S_i \cap \hat{D}_j \cong \mathbb{P}^1$.
- $E \cap S_1, E \cap S_2$ and $E \cap \hat{W}_0$ are three disjoint lines of one of the rulings on E .
- $E \cap \hat{D}_i$, $i = 1, 2, 3$ are the exceptional curves of $\hat{\pi} : \hat{D}_i \rightarrow D_i$ and give three disjoint lines on the other ruling of E .

See Figure 14 for a pictorial summary of this data.

Proof. This is largely a somewhat tedious calculation, so we merely summarize the most important points. First, from $x_0 = z^{(-1,-1,-1,-1)}$ and $x_4 = z^{(0,0,0,1)}$, one sees that the only exceptional divisors of $X_\Sigma \rightarrow X_{\Sigma_v}$ which both intersect X and on which the function $x_0 + x_4$ vanishes identically are those divisors corresponding to the rays generated by the points $(P_0 + P_4 + P_i)/3$, $i \in \{1, 2, 3\}$, (on which both x_0 and x_4 vanish to order one) and $(jP_0 + (3-j)P_4)/3$, $j \in \{1, 2\}$ (on which one of x_0, x_4 vanishes to order one and one to order two). The intersection of these five divisors with $\tilde{X} = \pi^{-1}(X)$ are the components D_i, S_j of $\tilde{w}^{-1}(0)$. Note that the toric divisor corresponding to $(P_0 + P_4 + P_i)/3$ maps surjectively

to the \mathbb{P}^1 stratum of X_{Σ_v} given by $x_0 = x_4 = u_i = s = 0$, and that D_i is the inverse image of the point defined by $u_j + u_k = 0$, $\{i, j, k\} = \{1, 2, 3\}$. One sees easily from Figure 13 that this fibre is isomorphic to \mathbb{P}^2 blown up at 3 points, as described in the statement of the lemma. Furthermore, the divisor corresponding to $(jP_0 + (3-j)P_4)/3$ maps surjectively to the \mathbb{P}^2 stratum of X_{Σ_v} given by $x_0 = x_4 = s = 0$, and S_j is the inverse image of the line $u_1 + u_2 + u_3 = 0$ under this map. Again, from Figure 13, it is not difficult to verify the description of S_j . Furthermore, again from the explicit subdivision given, one can verify the description of the intersections of the components D_1, D_2, D_3, S_1, S_2 .

To understand W_0 , one must compute the proper transform of $w^{-1}(0)$, which must be studied on open subsets corresponding to each triangle in Figure 13. We shall do this just for one crucial triangle, leaving it to the reader to check the others. Consider the triangle whose vertices are $(2P_0 + P_4)/3$, $(P_0 + 2P_4)/3$ and $(P_0 + P_4 + P_i)/3$, generating a cone τ in Σ . Without loss of generality, take $i = 1$. Note that τ^\vee is generated by $(-1, 0, -1, -1)$, $(0, 0, 1, 1)$, $(1, -1, 0, 0)$ and $\pm(0, -1, 1, 0)$.

Now on the open subset of X_{Σ_v} defined by the cone generated by P_0, P_1 and P_4 , u_2 and u_3 are non-zero. Thus on this open set, we can trivialize $-K_E$ by setting $u_2 = 1$. We get $u_1 = z^{(1,-1,0,0)}$, $u_3 = z^{(0,-1,1,0)}$ and $s = z^{(0,-1,0,0)}$. We can then take

$$\alpha_1 = z^{(-1,0,-1,-1)}, \quad \alpha_2 = z^{(0,0,1,1)}, \quad \alpha_3 = z^{(1,-1,0,0)}, \quad u_3^{\pm 1}$$

as coordinates on the open subset of X_Σ defined by τ . Note that the equation for $\tilde{X} = \pi^{-1}(X)$ is now

$$(7.3) \quad \alpha_3 + 1 + u_3 = \alpha_1 \alpha_2 \alpha_3$$

and the equation $x_0 + x_4 = 0$ becomes

$$\alpha_1^2 \alpha_2 \alpha_3 + \alpha_1 \alpha_2^2 \alpha_3 u_3^{-1} = 0.$$

We can factor out $\alpha_1 \alpha_2 \alpha_3$, reflecting that the divisors S_1, S_2 and D_1 occur in $\tilde{w}^{-1}(0)$ with multiplicity 1, leaving the equation for W_0 (multiplying by u_3 , keeping in mind it is invertible, and then eliminating this variable using (7.3)):

$$\alpha_1(\alpha_1 \alpha_2 \alpha_3 - \alpha_3 - 1) + \alpha_2 = 0.$$

One checks that this is non-singular, and intersects the other three irreducible components as claimed. Checking all charts, one finds the complete description of $\tilde{w}^{-1}(0)$ as given.

To describe $\hat{w}^{-1}(0)$, it is sufficient to note that the three divisors S_1, S_2 and W_0 meet each other mutually transversally along $S_1 \cap S_2$ in \tilde{X} . As a consequence, when one blows up the curve $S_1 \cap S_2$ in \tilde{X} , the proper transforms of these three divisors are now disjoint, and the exceptional divisor E , being a \mathbb{P}^1 -bundle over $S_1 \cap S_2$, now contains three disjoint sections, and hence is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The remaining details follow easily. \square

We also need a properification of the map $\tilde{w} : \tilde{X} \rightarrow \mathbb{C}$. To do so, begin by partially compactifying $\mathbb{A}^2 \times \mathbb{P}^3$ by embedding it in $\mathbb{A}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$, with coordinates $w, (y_0, y_1)$, on the \mathbb{A}^1 and \mathbb{P}^1 factors. This embedding is given by

$$(x_0, x_4, (u_1, u_2, u_3, s)) \mapsto (x_0 + x_4, (x_0, 1), (u_1, u_2, u_3, s)).$$

Then the closure X'_{Σ_v} of X_{Σ_v} in $\mathbb{A}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$ is given by the equation

$$y_0(y_1 w - y_0)u_1 u_2 u_3 = s^3 y_1^2.$$

Let X' be the hypersurface in X'_{Σ_v} defined by the equation $u_1 + u_2 + u_3 = s$.

- Lemma 7.2.** (1) *There is a resolution $\pi' : \tilde{X}' \rightarrow X'$ extending the resolution $\pi : \tilde{X} \rightarrow X$.*
- (2) *The map $\tilde{w} : \tilde{X} \rightarrow \mathbb{C}$ extends to a proper map $\tilde{w} : \tilde{X}' \rightarrow \mathbb{C}$.*
- (3) *$D := \tilde{X}' \setminus \tilde{X}$ is a normal crossings divisor, each component of which is mapped smoothly to \mathbb{C} under \tilde{w} .*
- (4) *Every fibre of \tilde{w} is a non-singular K3 surface except for $\tilde{w}^{-1}(0)$ and $\tilde{w}^{-1}(\pm 6\sqrt{3})$. The latter two fibres are K3 surfaces with one ordinary double point each.*
- (5) *$\tilde{w}^{-1}(0) = W'_0 \cup S_1 \cup S_2 \cup D_1 \cup D_2 \cup D_3$, where S_i, D_j are as in Lemma 7.1 and W'_0 is a compactification of W_0 .*

Proof. For (1), the open subset of X' where $y_0 = 1$ has the equation

$$(y_1 w - 1)u_1 u_2 u_3 = (u_1 + u_2 + u_3)^3 y_1^2$$

in $\mathbb{A}^2 \times \mathbb{P}^2$, and one finds the following singular locus. There are three curves of A_1 singularities given by $y_1 = u_i = u_j = 0$. There are also three curves of A_2 singularities, given by the equations $y_1 w - 1 = u_1 + u_2 + u_3 = u_i = 0$, $i \in \{1, 2, 3\}$. However, the latter curves are contained already in X , given here by $y_1 \neq 0$. Thus we may resolve X' by using the resolution $\pi \circ \hat{\pi}$ for X and in addition blowing up the three curves of A_1 singularities, which did not occur in X .

(2) is clear, since w agrees with the regular function w on X' , which is clearly proper. One then takes $\tilde{w} = w \circ \pi'$.

For (3), note that $X' \setminus X$ is given by setting $y_0 = 1, y_1 = 0$, giving the equation $u_1 u_2 u_3 = 0$ in $\mathbb{A}^1(w) \times \mathbb{P}^2$. Furthermore, π' blows up the curves $y_1 = u_i = u_j = 0$, and hence one sees easily that $\tilde{X}' \setminus \tilde{X}$ is $C_6 \times \mathbb{A}^1(w)$, where C_6 is a cycle of six rational curves. The restriction of \tilde{w} to this divisor is just given by projection onto $\mathbb{A}^1(w)$, making the result clear.

For (4), note that a fibre of $w : X' \rightarrow \mathbb{C}$ is given by fixing w , in which case this fibre can be described as the zero set of

$$y_0^2 u_1 u_2 u_3 - y_0 y_1 w u_1 u_2 u_3 + y_1^2 (u_1 + u_2 + u_3)^3$$

in $\mathbb{P}^1 \times \mathbb{P}^2$. The projection to \mathbb{P}^2 describes this surface as a partial resolution of a double cover of \mathbb{P}^2 , branched over the discriminant, the latter given by

$$u_1 u_2 u_3 (w^2 u_1 u_2 u_3 - 4(u_1 + u_2 + u_3)^3) = 0.$$

This is the union of the three coordinate lines and a (for general w) smooth cubic for which $u_i = 0$ is an inflectional tangent for each i . Suppose this cubic is indeed smooth. Then the double cover of \mathbb{P}^2 branched over this locus has three A_1 -singularities over the points $u_i = u_j = 0$ and three A_5 -singularities over the points $u_1 + u_2 + u_3 = u_i = 0$. However, the fibre of w as described in $\mathbb{P}^1 \times \mathbb{P}^2$ has partially resolved the A_5 -singularities, as the fibre of the projection to \mathbb{P}^2 over $u_1 + u_2 + u_3 = u_i = 0$ is a \mathbb{P}^1 , but the surface contains two A_2 -singularities along this \mathbb{P}^1 , at $y_0 = 0$ and $y_1 w - y_0 = 0$. The resolution $\tilde{X}' \rightarrow X'$ now resolves all remaining singularities minimally because it is crepant. Hence, a general fibre of \tilde{w} is a minimal K3 surface.

To identify the remaining singular fibres, we just need to know for what non-zero values of w the cubic $w^2 u_1 u_2 u_3 - 4(u_1 + u_2 + u_3)^3 = 0$ is singular. One checks easily that this occurs only when $w^2 = 4 \cdot 3^3$, at the point $(u_1, u_2, u_3) = (1, 1, 1)$. Furthermore, this point is a node of the cubic. This gives the two singular fibres with ordinary double points.

(5) is obvious. \square

The main result is then the following theorem:

Theorem 7.3. *Let $D = \tilde{X}' \setminus \tilde{X}$, $j^D : \tilde{X} \hookrightarrow \tilde{X}'$ the inclusion. Then*

$$\mathbb{H}^i(\phi_{\tilde{w}, \pm 6\sqrt{3}} \mathbf{R}j_*^D \mathbb{C}_{\tilde{X}}) = \begin{cases} \mathbb{C} & i = 2 \\ 0 & i \neq 2 \end{cases}$$

and

$$\mathbb{H}^i(\phi_{\tilde{w}, 0} \mathbf{R}j_*^D \mathbb{C}_{\tilde{X}}) = \begin{cases} \mathbb{C}^5 & i = 1 \\ \mathbb{C}^2 & i = 2 \\ \mathbb{C}^5 & i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $Y \subset \mathbb{P}^4$ is a smooth cubic three-fold, then

$$\dim_{\mathbb{C}} \bigoplus_{p \in \mathbb{C}} \mathbb{H}^i(\phi_{\tilde{w}, p} \mathbf{R}j_*^D \mathbb{C}_{\tilde{X}}) = \begin{cases} h^{1,2}(Y) = h^{2,1}(Y) & i = 1 \text{ or } 3 \\ \sum_{j=0}^3 h^{i,j}(Y) & i = 2. \end{cases}$$

Proof. The description of D in Lemma 7.2,(3) and Thm. 4.2 tell us that $R^q \phi_{\tilde{w}, p} \mathbf{R}j_*^D \mathbb{C}_{\tilde{X}}$ has support away from D , so we can instead calculate $R^q \phi_{\tilde{w}, p} \mathbb{C}_{\tilde{X}'}$. In the case that $p = \pm 6\sqrt{3}$, we immediately see the claimed result from Lemma 7.2,(4).

We can now describe $\mathcal{F} = \phi_{\bar{w},0}\mathbb{C}_{\tilde{X}'}$ as follows¹². We choose a retraction map $r : \tilde{X}'_t \rightarrow \tilde{X}'_0$ for $t \in \mathbb{C}$ close to 0. Then $\mathcal{F} = \text{Cone}_M(\mathbb{C}_{\tilde{X}'_0} \rightarrow Rr_*\mathbb{C}_{\tilde{X}'_t})$, where $\mathbb{C}_{\tilde{X}'_0} \rightarrow Rr_*\mathbb{C}_{\tilde{X}'_t}$ is the canonical map. We wish to compute the hypercohomology of \mathcal{F} . To describe \mathcal{F} , let $\hat{X}' \rightarrow \tilde{X}'$ be the blowup along ℓ , as in Lemma 7.1, coming with the potential $\hat{w} : \hat{X}' \rightarrow \mathbb{C}$. We note first that we can choose r as a composition $\hat{r} : \hat{X}'_t \rightarrow \hat{X}'_0$ and the blow-down $\hat{X}'_0 \rightarrow \tilde{X}'_0$. Further, we can make a base-change $\hat{X}' \times_{\mathbb{C}} \mathbb{C}$ via the map $\mathbb{C} \rightarrow \mathbb{C}$ given by $w \mapsto w^3$, and then normalizing. This produces a family $\bar{w} : \bar{X}' \rightarrow \mathbb{C}$ where now \bar{X}' has quotient singularities (what is called a V -manifold in the literature). The effects of this on the central fibre are easily seen to be as follows. One has

$$\bar{w}^{-1}(0) = W'_0 \cup S_1 \cup S_2 \cup \hat{D}_1 \cup \hat{D}_2 \cup \hat{D}_3 \cup \bar{E},$$

where the first six divisors are identical to those appearing in \hat{X}'_0 , and the last one is a cyclic triple cover of E totally ramified over $E \cap (W'_0 \cup S_1 \cup S_2 \cup \hat{D}_1 \cup \hat{D}_2 \cup \hat{D}_3)$.

This allows us to choose \hat{r} as a composition of a retraction $\bar{r} : \bar{X}'_t \rightarrow \bar{X}'_0$ followed by the projection $\bar{X}'_0 \rightarrow \hat{X}'_0$. The advantage of working with \bar{r} is that this is relatively easy to understand. One can in fact use techniques from log geometry, namely the Kato-Nakayama construction.

Given a log analytic space (X, \mathcal{M}_X) , there is a topological space X_{\log} along with a continuous map $\rho : X_{\log} \rightarrow X$. To define X_{\log} , we define a log point $P := (\text{Spec } \mathbb{C}, \mathbb{R}_{\geq 0} \times S^1)$, where the structure map $\alpha : \mathbb{R}_{\geq 0} \times S^1 \rightarrow \mathbb{C}$ is given by $\alpha(h, e^{i\theta}) = he^{i\theta} \in \mathbb{C}$. Then as a set, $X_{\log} = \text{Hom}(P, X)$. There is an obvious map $\rho : X_{\log} \rightarrow X$ taking a morphism $P \rightarrow X$ to its image. There is a natural topology on X_{\log} . If $f : X \rightarrow Y$ is a morphism of log schemes, there is the obvious map $f_{\log} : X_{\log} \rightarrow Y_{\log}$, which is also continuous, so the construction is functorial. See [KN99], [NO10] for details.

We use this as follows. If one puts the divisorial log structure on \bar{X}' given by the divisor $\bar{X}'_0 \subseteq \bar{X}'$, and on \mathbb{C} the divisorial log structure given by $0 \in \mathbb{C}$, the map $\bar{w} : \bar{X}' \rightarrow \mathbb{C}$ becomes log smooth in a neighbourhood of \bar{X}'_0 . The space \mathbb{C}_{\log} is just the real oriented blowup of \mathbb{C} at the origin. We then have a diagram

$$\begin{array}{ccc} \bar{X}'_{\log} & \xrightarrow{\rho'} & \bar{X}' \\ \bar{w}_{\log} \downarrow & & \downarrow \bar{w} \\ \mathbb{C}_{\log} & \xrightarrow{\rho} & \mathbb{C} \end{array}$$

The map ρ' defines a homeomorphism $(\rho')^{-1}(\bar{X}' \setminus \bar{X}'_0) \rightarrow \bar{X}' \setminus \bar{X}'_0$ because the log structure is only non-trivial on \bar{X}'_0 . Furthermore, if U is an open neighbourhood of $0 \in \mathbb{C}$ which does not contain any non-zero critical values of \bar{w} , then the restriction of \bar{w}_{\log} to $\bar{w}_{\log}^{-1}\rho^{-1}(U)$ is a topological fibre bundle by [NO10]. In particular, all fibres of \bar{w}_{\log} over $\rho^{-1}(U)$ are

¹²We omit the customary shift [1] here.

homeomorphic. For $t \in \rho^{-1}(U \setminus \{0\})$, $\overline{w}_{\log}^{-1}(t)$ is homeomorphic via ρ' to $\overline{X}'_{\rho(t)}$, while for $t_0 \in \rho^{-1}(0)$, we can take the map

$$\rho'|_{\overline{w}_{\log}^{-1}(t_0)} : \overline{w}_{\log}^{-1}(t_0) \rightarrow \overline{X}'_0$$

to be the desired retraction \bar{r} .

The advantage of this description is that it is now easy to describe the topology of this retraction. Indeed, \bar{r} is an isomorphism over the smooth points of \overline{X}'_0 , has fibre S^1 over the double points of \overline{X}'_0 , and fibre $S^1 \times S^1$ over the triple points of \overline{X}'_0 . The inverse image under \bar{r} of an irreducible component F of \overline{X}'_0 can be viewed as the real oriented blow-up of $\text{Sing}(\overline{X}'_0) \cap F$ inside F . Taking r to be the composition of \bar{r} with the projections $\overline{X}'_0 \rightarrow \widehat{X}'_0 \rightarrow \tilde{X}'_0$, we can describe the fibres of r as follows.

First, r is an isomorphism away from $\text{Sing}(\tilde{X}'_0)$. The fibre of r over a double point of \tilde{X}'_0 is S^1 . Next, we have the set $\ell = S_1 \cap S_2 \cap W_0 \subseteq \tilde{X}'_0$, a copy of \mathbb{P}^1 , with three special points $Q_i = \ell \cap D_i$, $i = 1, 2, 3$. Then the fibre of r over a point of $\ell^o = \ell \setminus \{Q_1, Q_2, Q_3\}$ can be described as follows. We have $\overline{E} \subseteq \overline{X}'_0$, and the projection to \tilde{X}'_0 yields an elliptically fibred K3 surface $f : \overline{E} \rightarrow \ell$. A fibre over a point of ℓ^o is a triple cover of \mathbb{P}^1 branched at three points. Since $\bar{r}^{-1}(\overline{E})$ is the real oriented blow-up of \overline{E} along the ramification locus of the projection $\overline{E} \rightarrow \widehat{E}$, one sees that the fibre of r over a point of ℓ^o is the real blow-up of an elliptic curve at three points. Finally, $r^{-1}(Q_i)$ can be described from this point of view as $S^1 \times M$, where M is a real blow-up of a \mathbb{P}^1 at three points.

Given this description, one can describe $R^1 r_* \mathbb{C}$ and $R^2 r_* \mathbb{C}$ as follows. Note that for a point $x \in \ell^o$, $H^1(r^{-1}(x), \mathbb{C}) \cong \mathbb{C}^4$, with \mathbb{C}^2 coming from the image of $H^1(T^2, \mathbb{C})$ under the embedding $r^{-1}(x) \hookrightarrow T^2$. The other \mathbb{C}^2 comes from removing three disks. As x varies, one can then describe $R^1 r_* \mathbb{C}|_{\ell^o} = R^1 f_* \mathbb{C} \oplus \mathbb{C}^2$. Note that because of monodromy in the elliptic fibration f , the pushforward of $R^1 f_* \mathbb{C}|_{\ell^o}$ across ℓ is just the extension by zero of $R^1 f_* \mathbb{C}|_{\ell^o}$. From this, one finds one can write

$$R^1 r_* \mathbb{C} = \mathcal{G} \oplus R^1 f_* \mathbb{C},$$

where \mathcal{G} has stalks \mathbb{C}^2 on ℓ^o , \mathbb{C} at all double points of \tilde{X}'_0 , and stalk $H^1(S^1 \times M, \mathbb{C}) = \mathbb{C}^3$ at Q_1, Q_2, Q_3 . The sheaf \mathcal{G} is constant on each connected component of $\text{Sing}(\tilde{X}'_0) \setminus \{Q_1, Q_2, Q_3\}$, and the generization maps on stalks $\mathcal{G}_{Q_i} \rightarrow \mathcal{G}_\eta$ for η a generic point of $\text{Sing}(\tilde{X}'_0)$ are seen to be surjective. The sheaf $R^2 r_* \mathbb{C}$ is much easier to describe: it is supported on the set $\{Q_1, Q_2, Q_3\}$ with stalk $H^2(M \times S^1) \cong \mathbb{C}^2$ at each of these points.

It is then not difficult to work out the E_2 term for the hypercohomology spectral sequence $E_2^{pq} = H^q(\mathcal{H}^p(\mathcal{F})) \Rightarrow \mathbb{H}^n(\mathcal{F})$. Indeed, $H^0(R^1 r_* \mathbb{C}) = H^0(\mathcal{G}) = \mathbb{C}^5$, since a section of \mathcal{G} is entirely determined by its stalks at Q_1, Q_2, Q_3 , and the generization of these stalks at the

generic point of ℓ must agree. Then we have an exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \mathbb{C}_\ell^2 \oplus \bigoplus_{i=1}^9 \mathbb{C}_{d_i} \rightarrow \bigoplus_{i=1}^3 \mathbb{C}_{Q_i}^2 \rightarrow 0,$$

where d_1, \dots, d_9 are the closures of the irreducible components of the double point locus, and \mathbb{C}_S denotes the constant sheaf \mathbb{C} on the variety S . From $H^0(\mathcal{G}) = \mathbb{C}^5$, one obtains from the long exact cohomology sequence that $H^1(\mathcal{G}) = 0$ and $H^2(\mathcal{G}) \cong \mathbb{C}^{11}$. Finally, one can check from the Leray spectral sequence for $f : E \rightarrow \ell$ and the fact that E is a singular K3 surface with 9 A_2 -singularities that $H^1(R^1 f_* \mathbb{C}) = \mathbb{C}^2$ and $H^2(R^1 f_* \mathbb{C}) = 0$. Putting this together, we obtain the E_2 term

$$\begin{array}{ccccc} H^0(R^2 r_* \mathbb{C}) = \mathbb{C}^6 & & 0 & & 0 \\ & \searrow d & & & \\ H^0(R^1 r_* \mathbb{C}) = \mathbb{C}^5 & & H^1(R^1 r_* \mathbb{C}) = \mathbb{C}^2 & \rightarrow & H^2(R^1 r_* \mathbb{C}) = \mathbb{C}^{11} \\ & & 0 & & 0 \end{array}$$

Finally, we need to show the map d is injective. First note that this map coincides with the same map in the Leray spectral sequence for r , and hence $\ker d$ is the image of the natural map $H^2(\tilde{X}_t, \mathbb{C}) \rightarrow H^0(R^2 r_* \mathbb{C})$. This map is dual to the natural map

$$(7.4) \quad \bigoplus_{i=1}^3 H_2(r^{-1}(Q_i), \mathbb{C}) \rightarrow H_2(\tilde{X}_t, \mathbb{C}).$$

But $H_2(r^{-1}(Q_i), \mathbb{C})$ is generated by the connected components of the boundary of $r^{-1}(Q_i)$ (with one relation), and it is easy to see that these cycles are bounded by the closure of the sets $r^{-1}(d_i \setminus \{Q_1, Q_2, Q_3\})$, for various i . Thus the map (7.4) is zero, from which we conclude that $\ker d = 0$. So the $E_3 = E_\infty$ term is

$$\begin{array}{ccc} 0 & 0 & 0 \\ \mathbb{C}^5 & \mathbb{C}^2 & \mathbb{C}^5 \\ 0 & 0 & 0 \end{array}$$

This shows the remaining claims, with the cohomology of a cubic threefold being well-known. \square

It has been conjectured in [Ka10] and verified along a list of cases that a three-dimensional Fano manifold is non-rational if its resolved mirror dual has a fibre with non-isolated singularities and non-unipotent monodromy. See also [KP09]. A general cubic being non-rational by a theorem of Clemens-Griffiths [CG72], we verify this conjecture for the cubic by computing the monodromy in cohomology of the family $\tilde{w} : \tilde{X}' \rightarrow \mathbb{A}^1$ around the zero fibre.

Proposition 7.4. *Let $t_0 \in \mathbb{C}$ be a point near 0, and let $T : H^2(\tilde{w}^{-1}(t_0), \mathbb{C}) \rightarrow H^2(\tilde{w}^{-1}(t_0), \mathbb{C})$ be the monodromy operator associated to a counter-clockwise loop around the origin based at t_0 . Then $T^3 = I$, so that the eigenvalues of T are third roots of unity. Furthermore, eigenvalue 1 has multiplicity 20, and the two primitive third roots of unity each have multiplicity one.*

Proof. Let $\bar{w} : \bar{X}' \rightarrow \mathbb{C}$ be as in the proof of Theorem 7.3, and let $Y = \bar{w}^{-1}(0)$. Then \bar{X}' is a V -manifold and Y is a V -normal crossings divisor in \bar{X}' . Recall Y has seven irreducible components, six isomorphic to $W'_0, S_1, S_2, D_1, D_2, D_3$ respectively, and the seventh being \bar{E} a cyclic triple cover of $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ totally ramified over a union of six lines, three in each ruling. Note \bar{E} is a K3 surface with 9 A_2 -singularities, and thus $H^2(\bar{E}, \mathbb{C}) \cong \mathbb{C}^4$, while $H^0(\bar{E}, \mathbb{C}) \cong H^4(\bar{E}, \mathbb{C}) \cong \mathbb{C}$.

Since all components of Y are reduced, the monodromy $\tilde{T} = T^3$ of $\bar{X}' \rightarrow \mathbb{A}^1$ around the origin must be unipotent, but since Y contains a K3 component, $\bar{X}' \rightarrow \mathbb{A}^1$ must be birationally equivalent to a type I degeneration of K3 surfaces. Thus $\tilde{T} = I$, so we see $T^3 = I$.

Let $\lambda : Y \rightarrow Y$ be a generator of the \mathbb{Z}_3 -action on Y arising from the construction of Y as the normalization of a cyclic triple cover. Then λ acts component-wise on Y , hence acts on each Y^r , and according to the proof of (2.13) of [St76], the action of λ^* on the weight spectral sequence

$$E_1^{-r, q+r} = \bigoplus_{k \geq 0, -r} H^{q-r-2k}(Y^{2k+r+1}, \mathbb{C})(-r-k) \Rightarrow H^q(\tilde{w}^{-1}(t_0), \mathbb{C})$$

coincides with the action of T . However, one sees easily that this action is trivial except for the action on the contribution $H^2(\bar{E}, \mathbb{C})$, which appears in $E_1^{0,2}$. Since the quotient of \bar{E} by the action the \mathbb{Z}_3 -action is $\mathbb{P}^1 \times \mathbb{P}^1$, the action of λ^* on $H^2(\bar{E}, \mathbb{C})$ must have only a two-dimensional invariant subspace, hence λ^* has in addition two eigenvalues being primitive third roots of unity, each appearing with multiplicity one. From this one concludes the same is true for the action of T on $H^2(\tilde{w}^{-1}(t_0), \mathbb{C})$. \square

APPENDIX A. A BINOMIAL IDENTITY

We include the proof of a binomial identity which we use in §3 and §5.

Proposition A.1. *For $n, k, m, p \in \mathbb{Z}_{\geq 0}$, we have*

(1)

$$\binom{n}{k} = (-1)^m \sum_{i \geq 0} (-1)^i \binom{i}{m} \binom{n+m+1}{k+1+i}$$

(2) (*Vandermonde's identity*)

$$\binom{m+n}{k} = \sum_{i \geq 0} \binom{m}{i} \binom{n}{k-i}$$

Proof. (2) is standard. To see (1), starting with $\binom{n}{k}$ and $i = 0$, the iterated insertion of $\binom{n}{k+i} = \binom{n+1}{k+1+i} - \binom{n}{k+1+i}$ yields $\binom{n}{k} = \sum_{i \geq 0} (-1)^i \binom{n+1}{k+1+i}$. This being the base case for $m = 0$, the general case follows by induction by inserting the base case in the induction hypothesis; indeed,

$$\begin{aligned} & (-1)^m \sum_{i \geq 0} (-1)^i \binom{i}{m} \binom{n+m+1}{k+1+i} \\ &= (-1)^m \sum_{i \geq 0} (-1)^i \binom{i}{m} \sum_{j \geq 0} (-1)^j \binom{n+m+2}{k+2+i+j} \\ &= (-1)^{m+1} \sum_{i'=1+i+j \geq 0} (-1)^{i'} \sum_{j \geq 0} \binom{i'-j-1}{m} \binom{n+(m+1)+1}{k+1+i'}, \end{aligned}$$

from which the assertion follows by noting that $\sum_{j=0}^{i'-1} \binom{i'-j-1}{m} = \binom{i'}{m+1}$. \square

REFERENCES

- [AAK] M. Abouzaid; D. Auroux; L. Katzarkov: “Mirror symmetry for blowups and hypersurfaces in toric varieties”, in preparation.
- [AFOO] M. Abouzaid; K. Fukaya; Y.-G. Oh; H. Ohta, K. Ono: “Quantum cohomology and split generation in Lagrangian Floer theory”, in preparation.
- [Ab10] M. Abouzaid: “A geometric criterion for generating the Fukaya category”, Publ. Math. Inst. Hautes Études Sci. No. **112** (2010), 191–240.
- [Aur07] D. Auroux, *Mirror symmetry and T-duality in the complement of the anticanonical divisor*, *J. Gökova Geom. Topol.* **1**, 2007, p.51–91.
- [Bat94] V. Batyrev: “Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties”, *J. Algebraic Geom.* **3**, 1994, p.493–535.
- [BB94] V. Batyrev, L. Borisov: “On Calabi-Yau Complete Intersections in Toric Varieties”, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, p.39–65.
- [BB96] V. Batyrev, L. Borisov: “Mirror duality and string-theoretic Hodge numbers”, *Invent. Math.* **126**(1), 1996, p.183–203.
- [Cl08] P. Clarke: “Duality for toric Landau-Ginzburg models”, arXiv:0803.0447.

- [CPS11] M. Carl, M. Pumperla, B. Siebert: “A tropical view on Landau-Ginzburg models”, Siebert’s webpage.
- [CG72] H. Clemens, P. Griffiths: “The intermediate Jacobian of the cubic threefold”, *Annals of Mathematics* Second Series **95**(2), 1972, p.281–356.
- [DK86] V.I. Danilov, A.G. Khovanskii: “Newton polyhedra and an algorithm for computing Hodge-Deligne numbers”, *Izv. Akad. Nauk SSSR Ser. Mat.* **50:5**, 1986, p.925–945.
- [Del73] P. Deligne: “Le formalisme des cycles évanescents”, *SGA 7-II*, Exp. XIII et XIV, Lect. Notes in Math. **340**, 1973, p.82–164.
- [DelTH] P. Deligne: “Théorie de Hodge II” / “Théorie de Hodge III”, *Inst. Hautes Études Sci. Publ. Math.*, **40**, 1971, p.5–57 / **44**, 1974, p.5–77.
- [Ef09] A.I. Efimov: “Homological mirror symmetry for curves of higher genus”, math.AG/0907.3903v4, 42.pp.
- [FOOO] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono: “Lagrangian Floer theory and mirror symmetry on compact toric manifolds,” 2010, arXiv:1009.1648.
- [Ga] S. Ganatra: “Symplectic cohomology and duality for the wrapped Fukaya category”, in prep.
- [Gi96] A. Givental: “A mirror theorem for toric complete intersections” in Topological field theory, primitive forms and related topics (Kyoto, 1996), *Progr. Math.*, **160**, Birkhäuser Boston, Boston, MA, 1998, p.141–175,
- [Gr10] M. Gross: “Mirror symmetry for \mathbb{P}^2 and tropical geometry”, *Adv. Math.* **224**(1) (2010), 169–245.
- [GS03] M. Gross, B. Siebert: “Mirror symmetry via logarithmic degeneration data I”, *Journal Of Differential Geometry*, **72**, (2006) : p.169–338.
- [GS10] M. Gross, B. Siebert: “Mirror symmetry via logarithmic degeneration data II”, *J. Algebraic Geom.* **19** (2010), p.679–780.
- [GS11] M. Gross, B. Siebert: “From real affine geometry to complex geometry”, *Annals of Math.*, **174**, p.1301–1428.
- [HW09] M. Herbst, J. Walcher: “On the unipotence of autoequivalences of toric complete intersection Calabi-Yau categories”, *Math. Annalen*, 29 July 2011, p. 1-20. doi:10.1007/s00208-011-0704-x.
- [HV00] K. Hori, C. Vafa: “Mirror symmetry”, 2000, hep-th/0002222.
- [ILP11] N. Ilten, J. Lewis, V. Przyjalkowski: “Toric degenerations of Fano threefolds giving weak Landau-Ginzburg models”, math.AG/1102.4664v2, 22p.
- [Is10] M. Isik: “Equivalence of the Derived Category of a Variety with a Singularity Category”, arXiv:math/1011.1484
- [KS10] E. Katz, A. Stapledon: “Tropical Geometry and the Motivic Nearby Fiber”, *Compositio Math.*, to appear.
- [KN99] K. Kato, C. Nakayama: “Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over \mathbb{C} ”, *Kodai Math. J.*, **22**, 1999, p.161–186.
- [KP09] L. Katzarkov, V. Przyjalkowski: “Generalized Homological Mirror Symmetry and cubics” *Proc. Steklov Inst. Math.* **264**(1), 2009, p.87–95.
- [KKP08] L. Katzarkov, M. Kontsevich, T. Pantev: “Hodge theoretic aspects of mirror symmetry”, *From Hodge theory to integrability and TQFT tt*-geometry*, p.7–174, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.
- [Ka10] L. Katzarkov: “Generalized Homological Mirror Symmetry and Rationality questions” in “Cohomological and Geometric Approaches to Rationality Problems - New Perspectives”, *Progress in Mathematics* **242**, Birkhäuser, ed. Bogomolov, Tschinkel, 1st ed., 2010, p.163–208.

- [KKOY09] A. Kapustin, L. Katzarkov, D. Orlov, M. Yotov: “Homological Mirror Symmetry for manifolds of general type”, *Cent.Eur.J.Math.*, **7**(4), 2009, p.571–605.
- [Ko94] M. Kontsevich: “Homological algebra of mirror symmetry”, International Congress of Mathematicians, Zürich, Switzerland, 3 - 11 Aug 1994, 120–139, math.AG/9411018
- [LP11] K.H. Lin, D. Pomerleano: “Global matrix factorizations”, math.AG/1101.5847v1, 15p.
- [NO10] C. Nakayama, A. Ogus: “Relative rounding in toric and logarithmic geometry,”, *Geom. Topol.*, **14**, 2010, p.2189–2241.
- [NS06] T. Nishinou, B. Siebert: “Toric degenerations of toric varieties and tropical curves,” *Duke Math. J.*, **135** (2006), 1–51.
- [OV07] A. Ogus, V. Vologodsky: “Nonabelian Hodge Theory in Characteristic p”, Publications Mathématiques de l’I.H.E.S. **106**, (2007).
- [Or05] D. Orlov: “Triangulated categories of singularities and equivalences between Landau-Ginzburg models”, *Sb. Math.*, **197**(12), 2006, p.1827–1840, math.AG/0503630.
- [Or11] D. Orlov: “Matrix factorizations for nonaffine LG models”, *Tr. Mat. Inst. Steklova*, **246**(Algebr. Geom. Metody, Svyazi i Prilozh.), 2004, p.240–262.
- [PS08] C.A.M. Peters, J.H.M. Steenbrink, “Mixed Hodge Structures”, Springer, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3. Folge, A Series of Modern Surveys in Mathematics, **52**, 2008.
- [Rud10] H. Ruddat: “Log Hodge groups on a toric Calabi-Yau degeneration”, in Mirror Symmetry and Tropical Geometry, *Contemp. Mathematics* **527**, AMS, Providence, RI, 2010, p. 113-164.
- [Sab99] C. Sabbah: “On a twisted de Rham complex”, *Tohoku Math. J.* **51**, (1999), p.125–140.
- [Sei01] P. Seidel: “Vanishing cycles and mutation”, European Congress of Mathematics, Vol. II (Barcelona, 2000), 65–85, Progr. Math., 202, Birkhäuser, Basel, 2001.
- [Sei07] P. Seidel: “Fukaya Categories and Picard-Lefschetz Theory”, *Zürich Lectures in Advanced Mathematics*, European Mathematical Society, 2008, 334 pages.
- [Sei08] P. Seidel: “Homological mirror symmetry for the genus two curve”, *J. Algebraic Geom.*, **20**, 2011, p.727–769.
- [Sh11] I. Shipman: “A Geometric Approach to Orlov’s Theorem”, arXiv:math/1012.5282
- [St75] J.H.M. Steenbrink: “Limits of Hodge Structures” *Invent. Math.*, **31**, p.229.
- [St76] J.H.M. Steenbrink: “Mixed Hodge structure on the vanishing cohomology” in *Real and complex singularities*, Oslo 1976. P. Holm ed. p. 525–563, Sijthoff-Noordhoff, Alphen a/d Rijn 1977
- [Th03] H. M. Thompson: “Comments on toric varieties”, arXiv:math/0310336

UCSD MATHEMATICS, 9500 GILMAN DRIVE, LA JOLLA, CA 92093-0112, USA

E-mail address: mgross@math.ucsd.edu

UNIVERSITÄT WIEN, FAKULTÄT FÜR MATHEMATIK, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA

UNIVERSITY OF MIAMI, DEPARTMENT OF MATHEMATICS, CORAL GABLES, FL 33146, USA

E-mail address: lkatzark@math.uci.edu

JGU MAINZ, INSTITUT FÜR MATHEMATIK, STAUDINGERWEG 9, 55099 MAINZ, GERMANY

E-mail address: ruddat@uni-mainz.de